An Introduction to Inequalities

N. L. Carothers

MATH 6820 Summer 2011

Bowling Green State University

Preface

These are notes for a six week summer course offered to graduate students at Bowling Green State University. The audience consisted primarily of first and second year students in pure mathematics, but also included a few students of applied mathematics and statistics.

I had no course outline or plan of attack other than to begin with the most widely known classical inequalities and see where that led us. For the most part I simply followed my nose and steered the discussion toward whatever held my interest at the moment. I should quickly add, however, that the presentation consisted of more than lectures: From the outset, the students were expected to present (and discuss) their solutions to the problem sets in class, which added immensely to the experience. On several occasions we devoted more than half the class period to discussing the problems. This material would easily lend itself to a student seminar, a course on problem solving, or even a course taught using the so-called Moore method.

I assumed very little by way of prerequisites and tried to make the course accessible to a wide audience, which may help to explain the selection of topics. There are a few topics that I would like to have covered (topics that required a knowledge of Lebesgue integration, for example, or rudimentary functional analysis) that I feared would be beyond the grasp of some of the students. Still, I think I managed to demonstrate a variety of techniques and methods of proof in this brief survey, although my personal preferences, which lean toward applications of convexity (and, more generally, toward an analytic rather than algebraic perspective) are plainly reflected in these notes.

Preliminaries

Obviously, the comparison of quantities is an essential tool in mathematics. In this course we'll be concerned with all manner of inequalities and, in particular, their systematic study and the tools used in their creation.

The *art* of inequalities is found in the clever, often subtle methods used to generate and verify them. The *science* of inequalities lies in their careful interpretation and in the knowledge of their scope and limitations.

We will take great pains to distinguish between strict inequalities ($<$) and weak inequalities ($\leq$). In the case of weak inequalities, we will carefully examine the case for equality. While we will freely use techniques from calculus, we will be sparing with *limits*, lest strict inequalities turn into weak inequalities.

Although the study of inequalities is arguably a subfield of real analysis, we will occasionally find recourse to tools from complex analysis, linear algebra, geometry, and, of course, algebra. Moreover, we will uncover applications to all of these fields (and more!).

As a starting point, we'll begin with two simple axioms:

(1) A given real number a satisfies precisely one of the following: $a < 0$, $a = 0$, $a > 0$.

(2) If $a > 0$ and $b > 0$, then $ab > 0$ and $a + b > 0$.

There are obvious variations and extensions of these axioms; for example:

(1′) Given a, $b \in \mathbb{R}$, precisely one of the following holds: $a < b$, $a = b$, $a > b$.

and:

Theorem. *If $a > b > 0$ and $c > d > 0$, then $ac > bd$.*

Proof. Given $a > b$, we have $a - b > 0$ and, hence, $(a-b)c > 0$. That is, $ac > bc$. Similarly, $c - d > 0$ leads to $b(c-d) > 0$ and, hence, $bc > bd$. Combining these two observations yields $ac > bd$. $\square$

Please consult the exercises for more "direct algebra" proofs.

As you can well imagine, we will also have frequent use of *mathematical induction.*

Theorem. *If $a > b > 0$, then $a^n > b^n$ for all $n = 1, 2, 3, \ldots$.*

Theorem. *$n! > 2^{n-1}$ for $n = 3, 4, \ldots$.*

Bernoulli's Inequality. *If $x \geq -1$, then $(1+x)^n \geq 1+nx$ for all $n = 1, 2, 3, \ldots$. Equality occurs only if $n = 1$ or $x = 0$.*

Proof. Fix $x \geq -1$. The inequality clearly holds for $n = 1$, so assume that $(1+x)^n \geq 1+nx$ for some $n > 1$. Then:

$$\begin{aligned}
(1+x)^{n+1} &= (1+x)(1+x)^n \\
&\geq (1+x)(1+nx) \quad \text{(because } 1+x \geq 0\text{)} \qquad (1) \\
&= 1 + (n+1)x + x^2 \\
&\geq 1 + (n+1)x. \qquad (2)
\end{aligned}$$

Thus, the inequality holds for all $n = 1, 2, 3, \ldots$. Note that equality in (2) forces $x = 0$. $\square$

Refinement. *If $x \geq -1$, $x \neq 0$, then $(1+x)^n > 1+nx$ for all $n = 2, 3, 4, \ldots$.*

Corollary. *Given x, $y > 0$, we have $(x+y)^n > x^n + nx^{n-1}y$ for all $n = 2, 3, 4, \ldots$. Equality can only occur if $y = 0$.*

Application. $\left(1 + \dfrac{1}{n}\right)^n$ *increases with n.*

Proof. It suffices to show that $\left(1+\frac{1}{n+1}\right)^{n+1} \Big/ \left(1+\frac{1}{n}\right)^n > 1$. For this we rewrite and apply Bernoulli's inequality:

$$\begin{aligned}
\frac{\left(1+\frac{1}{n+1}\right)^{n+1}}{\left(1+\frac{1}{n}\right)^n} &= \left(1+\frac{1}{n}\right) \cdot \left(\frac{1+\frac{1}{n+1}}{1+\frac{1}{n}}\right)^{n+1} \\
&= \left(1+\frac{1}{n}\right) \cdot \left(\frac{n^2+2n}{(n+1)^2}\right)^{n+1} \\
&= \left(1+\frac{1}{n}\right) \cdot \left(1-\frac{1}{(n+1)^2}\right)^{n+1} \\
&> \left(1+\frac{1}{n}\right) \cdot \left(1-\frac{1}{n+1}\right) = 1 \quad \text{(by Bernoulli).} \qquad \square
\end{aligned}$$

Example. $n! < \left(\frac{n}{2}\right)^n$ for $n \geq 6$.

Proof. It's not hard to check directly that $6! < 3^6$. For the inductive step:

$$\left(\frac{n+1}{2}\right)^{n+1} = \left(\frac{n+1}{2}\right)\left(\frac{n+1}{n}\right)^n\left(\frac{n}{2}\right)^n > \left(\frac{n+1}{2}\right)\cdot 2\cdot n! = (n+1)! \qquad \square$$

Bernoulli's inequality is deceptively powerful. As evidence of this, we'll use it to prove a classical inequality due to Cauchy.

The Arithmetic-Geometric Mean Inequality. *Given $n \in \mathbb{N}$ and positive numbers $a_1, a_2, \ldots, a_n$ we have*

$$\big(a_1 a_2 \cdots a_n\big)^{1/n} \;\leq\; \frac{a_1 + a_2 + \cdots + a_n}{n} \tag{AGM}$$

with equality if and only if $a_1 = a_2 = \cdots = a_n$.

Proof. We'll proceed by induction, of course. While the inequality is plainly true for $n = 1$, it's not hard to establish for $n = 2$ (see the exercises). So, suppose that the theorem holds for some fixed n and *all* choices of $a_1, a_2, \ldots, a_n > 0$. Given $a_{n+1} > 0$, let's suppose (for simplicity of notation) that $a_{n+1} \geq a_k$ for $k = 1, \ldots, n$ (otherwise, relabel the terms). Let's also agree to write

$$G_k = \big(a_1 a_2 \cdots a_k\big)^{1/k} \quad \text{and} \quad A_k = \frac{a_1 + a_2 + \cdots + a_k}{k}.$$

Then

$$A_{n+1} = \frac{nA_n + a_{n+1}}{n+1} = A_n + \frac{a_{n+1} - A_n}{n+1}$$

where, by assumption, $a_{n+1} \geq A_n$. Thus, by our Corollary to Bernoulli's inequality,

$$\begin{aligned} A_{n+1}^{n+1} &\geq A_n^{n+1} + (n+1)A_n^n\left(\frac{a_{n+1} - A_n}{n+1}\right) && (3)\\ &= a_{n+1}A_n^n \\ &\geq a_{n+1}G_n^n \;=\; G_{n+1}^{n+1}. && (4)\end{aligned}$$

Note that equality in (3) would force $a_{n+1} = A_n$ while equality in (4) would force $a_1 = \cdots = a_n$ (by hypothesis). Hence, equality throughout would yield $a_1 = a_2 = \cdots = a_{n+1}$. $\square$

The AGM is extremely important—and well worth examining in greater detail—including variations, extensions, and a number of alternate proofs.

Example. Amongst all rectangles having a fixed perimeter P, the square has maximum area.

Proof. Given a rectangle with sides of length a and b we have

$$\sqrt{A} = \sqrt{ab} \leq \frac{a+b}{2} = \frac{P}{4} \quad \text{(a constant).}$$

Thus, A maximum when $A = (P/4)^2$ which occurs when $a = b = P/4$. $\square$

This is a simplified example of what is sometimes called Dido's problem or the isoperimetric problem: Amongst all planar regions having a fixed perimeter, which one has maximum area? The answer (which we may have time to pursue later) is the circle.

Example. Find the dimensions of the most economical 12 ounce soda can.

Solution. This is a familiar problem from calculus: We want to find the right-circular cylinder having fixed volume $V = \pi r^2 h$ and *minimum* surface area $S = 2\pi r^2 + 2\pi r h$. But

$$\begin{aligned} S &= 2\pi r^2 + 2\pi r \cdot \frac{V}{\pi r^2} \\ &= 2\pi r^2 + \frac{2V}{r} \\ &= 2\pi r^2 + \frac{V}{r} + \frac{V}{r} && (5) \\ &\geq 3\left(2\pi r^2 \cdot \frac{V}{r} \cdot \frac{V}{r}\right)^{1/3} && (6) \\ &= 3\Big(2\pi V^2\Big)^{1/3}, \end{aligned}$$

where rewriting in (5) facilitates the application of the AGM in (6). Now equality occurs when $2\pi r^2 = V/r$; i.e., when $r = (V/2\pi)^{1/3}$. For this value of r we have

$$h = \frac{V}{\pi r^2} = \frac{V}{\pi} \cdot \left(\frac{2\pi}{V}\right)^{2/3} = 2\left(\frac{V}{2\pi}\right)^{1/3} = 2r.$$

In other words, the can should be as tall as it is wide. Not very realistic, but easy to solve.

Cauchy's original proof of the AGM used a technique called *backward induction*: Suppose that $P(n)$ is a proposition about the natural number n and suppose that

– $P(n)$ holds for *infinitely many* n, and

– $P(n-1)$ holds whenever $P(n)$ holds.

Then $P(n)$ holds for all n.

Cauchy's proof of the AGM. We first establish (AGM) for $n = 2^m$, $m = 1, 2, \ldots$. To begin, note that

$$a_1 a_2 = \left(\frac{a_1 + a_2}{2}\right)^2 - \left(\frac{a_1 - a_2}{2}\right)^2 < \left(\frac{a_1 + a_2}{2}\right)^2$$

unless $a_1 = a_2$. The inductive step will be clear once we progress from $m = 1$ to $m = 2$.

$$\begin{aligned} a_1 a_2 a_3 a_4 &\leq \left(\frac{a_1 + a_2}{2}\right)^2 \left(\frac{a_3 + a_4}{2}\right)^2 & (7) \\ &\leq \left(\frac{\frac{a_1+a_2}{2} + \frac{a_3+a_4}{2}}{2}\right)^{2\cdot 2} & (8) \\ &= \left(\frac{a_1 + a_2 + a_3 + a_4}{4}\right)^4. \end{aligned}$$

Equality in (7) forces $a_1 = a_2$ and $a_3 = a_4$, while equality in (8) forces $a_1 + a_2 = a_3 + a_4$; thus, we have strict inequality in either (7) or (8) unless $a_1 = a_2 = a_3 = a_4$. Continuing leads to

$$a_1 a_2 \cdots a_{2^m} < \left(\frac{a_1 + \cdots + a_{2^m}}{2^m}\right)^{2^m}$$

unless $a_1 = \cdots = a_{2^m}$. Now to see how the case of general n follows, consider this: Given $n < 2^m$ and $a_1, \ldots, a_n > 0$, set $b_1 = a_1, \ldots, b_n = a_n$, $b_{n+1} = \cdots = b_{2^m} = A_n$. Then

$$\begin{aligned} a_1 \cdots a_n A_n^{2^m - n} &= b_1 \cdots b_{2^m} \\ &< \left(\frac{b_1 + \cdots + b_{2^m}}{2^m}\right)^{2^m} & (9) \\ &= \left(\frac{nA_n + (2^m - n)A_n}{2^m}\right)^{2^m} = A_n^{2^m}. \end{aligned}$$

Hence, $a_1 \cdots a_n < A_n^n$. Note that equality in (9) would force $a_1 = \cdots = a_n$. □

As a corollary, we get a special case of Young's inequality, which we will see again very soon.

Corollary. *Let* x, $y > 0$ *and let* $1 \le m < n$. *Then* $x^{m/n} y^{1-m/n} \le \dfrac{mx + (n-m)y}{n}$. *That is, if* r *is a rational number satisfying* $0 < r < 1$, *then* $x^r y^{1-r} \le rx + (1-r)y$.

We conclude this section with a few stray facts from algebra and a few tools from calculus that will prove helpful in the sequel.

The Binomial Theorem. $(1+x)^n = \displaystyle\sum_{k=0}^{n} \binom{n}{k} x^k$.

Application. $\left(1 + \dfrac{1}{n}\right)^n < \displaystyle\sum_{k=0}^{n} \frac{1}{k!}$ for $n = 1, 2, \ldots$.

Proof. $\left(1 + \dfrac{1}{n}\right)^n = 1 + \displaystyle\sum_{k=1}^{n} \frac{n(n-1)\cdots(n-k+1)}{n^k \, k!} < \sum_{k=0}^{n} \frac{1}{k!}$. □

As it happens, $\displaystyle\sum_{k=0}^{n} \frac{1}{k!} < \left(1 + \frac{1}{n}\right)^{n+1}$, but this is harder to prove.

Geometric Series. For $r \ne 1$ we have $1 + r + r^2 + \cdots + r^n = \dfrac{1 - r^{n+1}}{1 - r}$. Thus, for $|r| < 1$, we have $\displaystyle\sum_{k=0}^{\infty} r^k = \frac{1}{1-r}$. (See the exercises for a proof that $r^n \to 0$ for $|r| < 1$.)

Application. $\displaystyle\sum_{k=0}^{n} \frac{1}{k!} < 3$ for all $n = 1, 2, \ldots$.

Proof. As we've seen, $k! \ge 2^{k-1}$ for $k \ge 2$. Hence,

$$\begin{aligned}
\sum_{k=0}^{n} \frac{1}{k!} &= 1 + 1 + \sum_{k=2}^{n} \frac{1}{k!} \\
&\le 1 + 1 + \sum_{k=2}^{n} \frac{1}{2^{k-1}} \\
&= 1 + \sum_{k=0}^{n-1} \left(\frac{1}{2}\right)^k \\
&< 1 + \frac{1}{1 - (1/2)} = 3. \quad \square
\end{aligned}$$

The Mean Value Theorem. *If* $f : I \to \mathbb{R}$ *is differentiable, then for* x, $y \in I$, *we have*

$$f(x) - f(y) = f'(c) \cdot (x - y)$$

for some c between x and y.

Application. For $x \geq -1$ and $r > 1$, we have $(1+x)^r \geq 1 + rx$ with equality only at $x = 0$.

Proof. If $f(x) = (1+x)^r$ for $x \geq -1$, then

$$(1+x)^r - 1 \;=\; f(x) - f(0) \;=\; r(1+c)^{r-1}x$$

for some c between x and 0. If $x > c > 0$, then $1 + c > 1$ and, hence, $(1+x)^r - 1 > rx$. If $-1 \leq x < c < 0$, then $0 < 1 + c < 1$ and, again, $(1+x)^r - 1 > rx$. $\square$

Application. $\left(1 + \dfrac{1}{x}\right)^x$ increases while $\left(1 + \dfrac{1}{x}\right)^{x+1}$ decreases for $x > 0$.

Proof. For $x > 0$, the mean value theorem assures us that $\log(x+1) - \log x = 1/c$ for some $x < c < x + 1$. Thus,

$$\begin{aligned} \frac{d}{dx}\Big\{x[\log(x+1) - \log x]\Big\} &= \log(x+1) - \log x - \frac{1}{x+1} \\ &= \frac{1}{c} - \frac{1}{x+1} > 0 \end{aligned}$$

and it follows that $\left(1 + \frac{1}{x}\right)^x = e^{x[\log(x+1) - \log x]}$ is increasing. Similarly,

$$\begin{aligned} \frac{d}{dx}\Big\{(x+1)[\log(x+1) - \log x]\Big\} &= \log(x+1) - \log x - \frac{1}{x} \\ &= \frac{1}{c} - \frac{1}{x} < 0, \end{aligned}$$

which means that $\left(1 + \frac{1}{x}\right)^{x+1} = e^{(x+1)[\log(x+1) - \log x]}$ is decreasing. $\square$

Here's a final application (whose proof is left as an exercise).

Application. If $x \neq y$ are positive, then

(i) $rx^{r-1}(x-y) > x^r - y^r > ry^{r-1}(x-y)$ for $r < 0$ or $r > 1$;

(ii) $rx^{r-1}(x-y) < x^r - y^r < ry^{r-1}(x-y)$ for $0 < r < 1$.

To see how this is related to our earlier work, watch closely:

$$\begin{aligned} & x^r - y^r < ry^{r-1}(x-y) \quad (0 < r < 1) \\ \Longrightarrow \quad & x^r < ry^{r-1}(x-y) + y^r \\ \Longrightarrow \quad & x^r y^{r-1} < r(x-y) + y = rx + (1-r)y. \end{aligned}$$

Look familiar?

Problem Set 1

1. If $a > b > 0$ and if p and q are positive integers, prove that $a^{p/q} > b^{p/q}$. In particular, given a, $b > 0$, conclude that $a > b$ if and only if $a^{p/q} > b^{p/q}$.

2. If $a \geq b > 0$, p is a nonnegative integer, and q is a positive integer, show that $a^{p/q} \geq b^{p/q}$, with equality occurring if and only if either (i) $a = b$ or (ii) $p = 0$.

3. If $a_1 \geq b_1 > 0$, $a_2 \geq b_2 > 0$, $\ldots$, $a_n \geq b_n > 0$, show that $a_1 a_2 \cdots a_n \geq b_1 b_2 \cdots b_n$. Moreover, equality holds if and only if $a_1 = b_1$, $a_2 = b_2$, $\ldots$, $a_n = b_n$.

4. If m and n are positive integers, show that $\sqrt{2}$ lies between m/n and $(m+2n)/(m+n)$. [Hint: Either $m/n < \sqrt{2}$ or $\sqrt{2} < m/n$.]

5. From the inequality $(a-b)^2 \geq 0$, deduce that $2ab \leq a^2 + b^2$, with equality occurring if and only if $a = b$.

6. Given a, $b > 0$, show that $\sqrt{ab} \leq (a+b)/2$, with equality if and only if $a = b$.

7. Amongst all rectangles having a fixed area A, prove that the square (with sides of length $\sqrt{A}$) has the smallest perimeter.

8. From the inequality $\left(\dfrac{1}{\sqrt{a}} - \dfrac{1}{\sqrt{b}}\right)^2 \geq 0$, deduce that

$$\frac{2}{(1/a) + (1/b)} \leq \sqrt{ab}$$

for all positive a, b. Under what circumstances does equality hold?

9. Show that $a + (1/a) \geq 2$ for all positive values of a. When does equality occur?

10. If $0 < c < 1$, use Bernoulli's inequality to show that $c^n \to 0$. [Hint: Write $c = 1/(1+x)$ where $x > 0$.]

11. If $c > 0$, use Bernoulli's inequality to show that $c^{1/n} \to 1$. [Hint: If $c > 1$, write $c^{1/n} = 1 + x_n$, where $x_n > 0$, and estimate x_n.]

12. Given $x > 0$, use Bernoulli's inequality to show that $(1 + (x/n))^n$ increases while $(1 + (x/n))^{n+1}$ decreases as n increases.

13. Given $a > 0$, show that $(1+a)^r > 1 + ra$ for any *rational* $r > 1$. [Hint: Write $r = p/q$, where $p > q$ are positive integers, and note that $(1+(x/p))^p \geq (1+(x/q))^q$ for $x > 0$.]

14. Prove by induction that $\left(\dfrac{n}{3}\right)^n < n! < \left(\dfrac{n}{2}\right)^n$ for $n \geq 6$.

15. Use the fact that $k! \geq 2^{k-1}$, for all $k = 1, 2, 3, \ldots$, to prove that $\displaystyle\sum_{k=0}^{n} \frac{1}{k!} < 3$ for all $n = 0, 1, 2, \ldots$.

16. Suppose that $0 \leq x_i \leq 1$ for $i = 1, 2, \ldots$. Prove that

$$(1 - x_1) \cdots (1 - x_n) \ \geq \ 1 - (x_1 + \cdots + x_n) + (x_1 x_2 + \cdots + x_{n-1} x_n)$$

for all $n = 2, 3, \ldots$.

17. For $n = 1, 2, 3, \ldots$, show that $2(\sqrt{n+1} - \sqrt{n}\,) < \dfrac{1}{\sqrt{n}} < 2(\sqrt{n} - \sqrt{n-1}\,)$ and use this to prove that, for $n = 2, 3, \ldots$,

$$2\sqrt{n} - 2 \ < \ \sum_{k=1}^{n} \frac{1}{\sqrt{k}} \ < \ 2\sqrt{n} - 1.$$

18. Prove that

$$\frac{1}{\sqrt{4n+1}} < \frac{1}{2} \cdot \frac{3}{4} \cdots \frac{2n-1}{2n} < \frac{1}{\sqrt{3n+1}}$$

for $n = 2, 3, 4, \ldots$.

Convex Functions

A function $f : I \to \mathbb{R}$, defined on a nontrivial interval I, is said to be *convex* if

$$f(\lambda x + \mu y) \le \lambda f(x) + \mu f(y) \tag{1}$$

whenever x, $y \in I$, λ, $\mu \ge 0$, $\lambda + \mu = 1$. If the inequality in (1) is always strict for $x \ne y$, λ, $\mu > 0$, we say that f is *strictly convex*. We say that f is *concave* (resp., *strictly concave*) if $-f$ is convex (resp., strictly convex).

Examples.

(1) $f(x) = ax + b$ is convex—and concave, too, of course.

(2) $f(x) = |x|$ is convex because $|\lambda x + \mu y| \le \lambda |x| + \mu |y|$ for λ, $\mu \ge 0$.

(3) $f(x) = x^2$ is strictly convex.

Proof. Because $\lambda + \mu = 1$ we have

$$\begin{aligned} \lambda x^2 + \mu y^2 - (\lambda x + \mu y)^2 &= \lambda\mu x^2 + \mu\lambda y^2 - 2\lambda\mu xy \\ &= \lambda\mu(x - y)^2 > 0, \end{aligned}$$

provided λ, $\mu > 0$ and $x \ne y$. $\square$

As it happens, the rather unassuming inequality in (1) actually implies that convex functions are quite well behaved. For example, it's not terribly hard to prove the following:

Fact. *Let $f : I \to \mathbb{R}$ be convex, where I is an open interval. Then*

(i) *f is* continuous *on I.*

(ii) *f is* differentiable *at all but (at most) countably many points of I.*

(iii) *f' is* continuous *wherever it exists.*

An equivalent characterization of convexity is provided by the "three chords lemma."

Theorem. *Let $f : I \to \mathbb{R}$. Then f is convex if and only if*

$$\frac{f(z)-f(x)}{z-x} \;\le\; \frac{f(y)-f(x)}{y-x} \;\le\; \frac{f(y)-f(z)}{y-z} \tag{2}$$

for all $x < z < y$ in I.

Proof. First suppose that f is convex. Given $x < z < y$, we can write

$$z \;=\; \frac{y-z}{y-x}\cdot x \;+\; \frac{z-x}{y-x}\cdot y$$

(where the coefficients of x and y are nonnegative and add to 1). Thus,

$$f(z) \;\le\; \frac{y-z}{y-x}\cdot f(x) \;+\; \frac{z-x}{y-x}\cdot f(y).$$

The inequalities in (2) now follow by rewriting; for example,

$$f(z)-f(x) \;\le\; \frac{z-x}{y-x}\cdot\big(f(y)-f(x)\big) \quad\Longrightarrow\quad \frac{f(z)-f(x)}{z-x} \;\le\; \frac{f(y)-f(x)}{y-x},$$

which is the first half of (1). The other half is entirely similar.

Now suppose that (2) holds. Take $x < y$ in I, λ, $\mu \ge 0$, $\lambda + \mu = 1$. Then $z = \lambda x + \mu y$ satisfies

$$z - x = \mu(y-x) \quad\text{and}\quad y - z = \lambda(y-x).$$

That is, $z \;=\; \dfrac{y-z}{y-x}\cdot x \;+\; \dfrac{z-x}{y-x}\cdot y$. Thus, from (2),

$$\begin{aligned} f(z) \;&\le\; f(x) \;+\; \frac{z-x}{y-x}\cdot\big(f(y)-f(x)\big) \\ &=\; f(x) + \mu\big(f(y)-f(x)\big) \\ &=\; \lambda f(x) + \mu f(y). \qquad \square \end{aligned}$$

Corollary. *If $f : (a,b) \to \mathbb{R}$ is* differentiable *and convex, then*

$$f'(x) \;\le\; \frac{f(y)-f(x)}{y-x} \;\le\; f'(y)$$

for all $x < y$ in (a,b). In particular, f' is increasing.

The converse of this result is also true and, indeed, well worth expanding on.

Theorem. *Let $f : (a, b) \to \mathbb{R}$ be differentiable. Then the following are equivalent:*

(i) *f is convex;*

(ii) *f' is increasing;*

(iii) *$f(x) + f'(x)(y - x) \leq f(y)$ for all $x < y$ in (a, b).*

Note that the left-hand side of (iii) is the tangent line to the graph of f at x. Thus, (iii) tells us that every tangent line to the graph lies *below* the graph. A similar result holds for nondifferentiable convex functions (see the exercises for details).

Proof. (i) implies (ii) is clear. (ii) implies (iii) follows from the mean value theorem. Indeed,

$$\begin{aligned} f(y) &= f(x) + f'(c)(y - x), \quad \text{where } x < c < y, \\ &\geq f(x) + f'(x)(y - x). \end{aligned}$$

Finally, (iii) implies (i). Given $x < z < y$ we have:

$$\begin{aligned} f(x) + f'(x)(z - x) &\leq f(z), \\ f(z) + f'(z)(x - z) &\leq f(x), \\ f(z) + f'(z)(y - z) &\leq f(y). \end{aligned}$$

Thus,

$$f'(x) \;\leq\; \frac{f(z) - f(x)}{z - x} \;\leq\; f'(z) \;\leq\; \frac{f(y) - f(z)}{y - z}.$$

As in the proof of the three chords lemma, writing $z = \lambda x + \mu y$ leads to $\lambda\big(f(z) - f(x)\big) \leq \mu\big(f(y) - f(z)\big)$ or $f(z) \leq \lambda f(x) + \mu f(y)$. $\square$

Claim. If $f : (a, b) \to \mathbb{R}$ is twice-differentiable, then f is convex if and only if $f'' \geq 0$ (that is, f is "concave up"). If $f'' > 0$, then f is strictly convex. (See the exercises.)

Examples.

(1) x^p is convex on $(0, \infty)$ for $p \geq 1$, and strictly convex for $p > 1$.

(2) e^x is strictly convex on $\mathbb{R}$.

(3) $\log x$ is strictly concave on $(0, \infty)$.

(4) $x \log x$ is strictly convex on $(0, \infty)$.

Applications to Classical Inequalities

We begin with an easy extension of our characterization of convex functions.

Jensen's Inequality. *$f : I \to \mathbb{R}$ is convex if and only if*

$$f(\lambda_1 x_1 + \cdots + \lambda_n x_n) \le \lambda_1 f(x_1) + \cdots + \lambda_n f(x_n)$$

whenever $x_1, \ldots, x_n \in I$, $\lambda_1, \ldots, \lambda_n \ge 0$, $\lambda_1 + \cdots + \lambda_n = 1$.

Jensen's characterization follows easily from ours by induction.

Armed with our knowledge of convex functions, we're ready for a fresh attack on certain classical inequalities. We begin with a generalization of the AGM (but please compare this result with Young's inequality).

Theorem. *Let $x_1, \ldots, x_k$, $\alpha_1, \ldots, \alpha_k > 0$ with $\alpha_1 + \cdots + \alpha_k = 1$. Then*

$$x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_k^{\alpha_k} \le \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_k x_k$$

with equality if and only if $x_1 = \cdots = x_k$.

Proof. Because $\log x$ is strictly concave on $(0, \infty)$ we have

$$\begin{aligned} \log\Big(x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_k^{\alpha_k}\Big) &= \alpha_1 \log x_1 + \alpha_2 \log x_2 + \cdots + \alpha_k \log x_k \\ &\le \log\big(\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_k x_k\big), \end{aligned}$$

with equality if and only if $x_1 = \cdots = x_k$. And, because $\log x$ (or e^x, if you prefer) is strictly increasing, the result follows. $\square$

The case $\alpha_1 = \cdots = \alpha_k = 1/k$ is the familiar AGM. Another special case is also familiar and will lead us to two new inequalities.

Young's Inequality. *If p, $q > 1$ satisfy $\frac{1}{p} + \frac{1}{q} = 1$, then $xy \le \dfrac{1}{p}\, x^p + \dfrac{1}{q}\, y^q$ for all x, $y > 0$, with equality if and only if $x^p = y^q$.*

This version of Young's inequality leads to a simple proof of:

Hölder's Inequality. *Let $x_1, \ldots, x_n,\ y_1, \ldots, y_n > 0$ and let $p,\ q > 1$ satisfy $\frac{1}{p} + \frac{1}{q} = 1$. Then*

$$\sum_{i=1}^{n} x_i y_i \ \le\ \left(\sum_{i=1}^{n} x_i^p\right)^{1/p} \left(\sum_{i=1}^{n} y_i^q\right)^{1/q} \tag{3}$$

with equality if and only if (x_i^p) and (y_i^q) are proportional*; that is, if and only if, for some α, β, not both zero, we have $\alpha x_i^p = \beta y_i^q$ for all $i = 1, \ldots, n$. The case $p = q = 2$ is usually referred to as the* Cauchy-Schwarz inequality.

Proof. We may assume that $u = \left(\sum_{i=1}^{n} x_i^p\right)^{1/p} \neq 0$ and $v = \left(\sum_{i=1}^{n} y_i^q\right)^{1/q} \neq 0$, in which case we have, from Young's inequality, that

$$\frac{x_i}{u} \cdot \frac{y_i}{v} \ \le\ \frac{1}{p}\left(\frac{x_i}{u}\right)^p + \frac{1}{q}\left(\frac{y_i}{v}\right)^q \tag{4}$$

for each i, with equality if and only if $(x_i/u)^p = (y_i/v)^q$. Summing over i, we get

$$\frac{1}{uv}\sum_{i=1}^{n} x_i y_i \ \le\ \frac{1}{pu^p}\sum_{i=1}^{n} x_i^p + \frac{1}{qv^q}\sum_{i=1}^{n} y_i^q \ =\ 1.$$

Thus, $\sum_{i=1}^{n} x_i y_i \le uv$. Please note that equality in (3) would force equality in (4) for all i and, hence, $(x_i/u)^p = (y_i/v)^q$ for all i; that is, (x_i^p) and (y_i^q) would be proportional. $\square$

Corollary. *Let $x_1, \ldots, x_n,\ y_1, \ldots, y_n > 0$, let $0 < p < 1$, and let q satisfy $\frac{1}{p} + \frac{1}{q} = 1$. Then*

$$\sum_{i=1}^{n} x_i y_i \ \ge\ \left(\sum_{i=1}^{n} x_i^p\right)^{1/p} \left(\sum_{i=1}^{n} y_i^q\right)^{1/q}$$

with equality if and only if (x_i^p) and (y_i^q) are proportional.

Proof. This version actually follows from our previous version after a bit of judicious rewriting:

$$\sum_{i=1}^{n} x_i^p \ =\ \sum_{i=1}^{n} (x_i y_i)^p \cdot y_i^{-p}. \tag{5}$$

Now we apply Hölder's inequality using the conjugate pair of exponents

$$p' = \frac{1}{p} > 1 \quad \text{and} \quad q' = -\frac{q}{p} = 1 - q > 1.$$

Note that

$$\frac{1}{p'} + \frac{1}{q'} = p + \left(-\frac{p}{q}\right) = p\left(1 - \frac{1}{p}\right) = 1.$$

Applying Hölder's inequality with this pair of exponents yields

$$\begin{aligned} \sum_{i=1}^n x_i^p \;&\le\; \left(\sum_{i=1}^n (x_i y_i)^{p\cdot(1/p)}\right)^p \left(\sum_{i=1}^n y_i^{(-p)\cdot(-q/p)}\right)^{-p/q} \\ &=\; \left(\sum_{i=1}^n x_i y_i\right)^p \left(\sum_{i=1}^n y_i^q\right)^{-p/q}. \end{aligned}$$

Taking p-th roots and rearranging the terms finishes the proof. $\square$

We will see a few more variations and extensions of Hölder's inequality. For now we'll settle for the following:

Theorem. *Let $a = (a_i)$, $b = (b_i), \ldots, h = (h_i)$ denote elements of $\mathbb{R}^n$ in which all entries are positive, let $\alpha, \beta, \ldots, \theta > 0$ with $\alpha + \beta + \cdots + \theta = 1$. Then*

$$\sum_{i=1}^n a_i^\alpha b_i^\beta \cdots h_i^\theta \;\le\; \left(\sum_{i=1}^n a_i\right)^\alpha \left(\sum_{i=1}^n b_i\right)^\beta \cdots \left(\sum_{i=1}^n h_i\right)^\theta$$

with equality if and only if $a, b, \ldots, h$ are all proportional.

The proof is essentially identical to the proof of Hölder's inequality for two sets of numbers: We apply Young's inequality (or the AGM) to the expression

$$\left(\frac{a_i}{\sum a}\right)^\alpha \left(\frac{b_i}{\sum b}\right)^\beta \cdots \left(\frac{h_i}{\sum h}\right)^\theta$$

and sum over i.

1. If $f : I \to \mathbb{R}$ is convex, show that $f(\lambda_1 x_1 + \cdots + \lambda_k x_k) \le \lambda_1 f(x_1) + \cdots + \lambda_k f(x_k)$, for any $x_1, \ldots, x_k \in I$ and any $\lambda_1, \ldots, \lambda_k > 0$ with $\lambda_1 + \cdots + \lambda_k = 1$. [This is sometimes called *Jensen's inequality*, and follows easily by induction on k.] If f is strictly convex, prove that equality occurs in Jensen's inequality (if and) only if all the x_i are equal.

2. Let $f, g : I \to \mathbb{R}$ be convex functions and let $\alpha \in \mathbb{R}$. Determine which of the following are convex functions: αf, $f + g$, $f - g$, $|f|$, fg, $\max\{f, g\}$, $\min\{f, g\}$.

3. Let $f, g : \mathbb{R} \to \mathbb{R}$ be convex functions with g increasing. Show that $g \circ f$ is convex. Deduce that if $h : \mathbb{R} \to \mathbb{R}$ is a positive function and if $\log h$ is convex, then h is convex. Give an example of a positive convex function on $\mathbb{R}$ whose logarithm is not convex.

4. Show that the reciprocal of a positive concave function is convex. Is the reciprocal of a positive convex function always concave?

5. Prove that $f : (0, \infty) \to \mathbb{R}$ is convex if and only if $g(x) = x f(1/x)$ is convex on $(0, \infty)$.

6. If f is convex on $(0, \infty)$, and if $x_1, \ldots, x_m, y_1, \ldots, y_m > 0$, show that

$$(x_1 + \cdots + x_m)\, f\left(\frac{y_1 + \cdots + y_m}{x_1 + \cdots + x_m}\right) \;\le\; x_1 f\left(\frac{y_1}{x_1}\right) + \cdots + x_m f\left(\frac{y_m}{x_m}\right).$$

Show that $f(x) = (1 + x^p)^{1/p}$ is convex on $(0, \infty)$ for $p \ge 1$, and deduce from the first part of the exercise that $\big[\,(x_1 + \cdots + x_m)^p + (y_1 + \cdots + y_m)^p\,\big]^{1/p} \le (x_1^p + y_1^p)^{1/p} + \cdots + (x_m^p + y_m^p)^{1/p}$.

7. Let $f : (a, b) \to \mathbb{R}$ be differentiable and convex. If $f'(x_0) = 0$, show that $f(x_0)$ is a global minimum for f.

8. Let $f : (a, b) \to \mathbb{R}$ be twice differentiable Show that f is convex if and only if $f'' \ge 0$. If $f'' > 0$, show that f is strictly convex. Give an example of a twice differentiable, strictly convex function for which f'' fails to be strictly positive.

9. If $f, g : I \to \mathbb{R}$ are nonnegative, increasing, and convex, then so is fg. [Hint: First note that $[f(x) - f(y)]\,[g(y) - g(x)] \le 0$ for $x < y$.]

10. Let $f : (a, b) \to \mathbb{R}$ be continuous. Show that f is convex if and only if, for all $a < s < t < b$, we have $(t - s)^{-1} \int_s^t f(x)\, dx \le [f(t) + f(s)]/2$.

11. Let $f : I \to \mathbb{R}$ be convex. Then f has left- and right-derivatives at each point a in the interior of I and, moreover, $f'_-(a) \le f'_+(a)$. If $a < b$ are points in the interior of I, show that

$$f'_+(a) \le \frac{f(b) - f(a)}{b - a} \le f'_-(b).$$

12. If $f : I \to \mathbb{R}$ is convex, then f is *locally Lipschitz* on I; that is, f is Lipschitz on every compact subinterval of I. In particular, f is continuous on I.

13. We say that $T(x) = mx+b$ is a *supporting line* to the graph of f at x_0 if $T(x_0) = f(x_0)$ and $T(x) \le f(x)$ for all x. Prove that $f : (a,b) \to \mathbb{R}$ is convex if and only if the graph of f has a supporting line at each $x_0 \in (a,b)$. [Hint: For the forward implication, take m with $f'_-(x_0) \le m \le f'_+(x_0)$.]

14. Prove that a convex function defined on a bounded interval is bounded below.

15. If $f : \mathbb{R} \to \mathbb{R}$ is convex and bounded above, prove that f is constant.

16. Given $f : I \to \mathbb{R}$, the *epigraph* of f is the set $\mathrm{epi} f = \{(x,y) : x \in I,\ y \ge f(x)\}$ in $\mathbb{R}^2$. Show that f is convex if and only if $\mathrm{epi} f$ is a convex subset of $\mathbb{R}^2$.

17. We say that $f : I \to \mathbb{R}$ is *lower semicontinuous* (l.s.c.) if, for each real α, the set $\{x \in I : f(x) \le \alpha\}$ is closed. Prove that a convex function $f : I \to \mathbb{R}$ is lower semicontinuous if and only if $\mathrm{epi} f$ is closed. For this reason, l.s.c convex functions are often referred to as *closed* convex functions.

10. Let $f : (a, b) \to \mathbb{R}$ be continuous. Show that f is convex if and only if, for all $a < s < t < b$, we have $(t-s)^{-1} \int_s^t f(x)\,dx \le [f(t) + f(s)]/2$.

Solution. ($\Longrightarrow$) If f is convex and if $a < s < t < b$, then, on the interval $[\,s, t\,]$, the graph of f lies below the chord joining $(s, f(s))$ and $(t, f(t))$; that is,

$$f(x) \le f(s) + \left(\frac{f(t) - f(s)}{t - s}\right)(x - s), \quad s \le x \le t.$$

Integrating both sides over $[\,s, t\,]$ yields:

$$\int_s^t f(x)\,dx \le f(s)(t-s) + \frac{1}{2}\left(\frac{f(t) - f(s)}{t - s}\right)(t-s)^2 = \left(\frac{f(s) + f(t)}{2}\right)(t - s).$$

($\Longleftarrow$) To prove the converse, it suffices to show that if f is not convex, then, on some subinterval $[\,s, t\,] \subset (a, b)$, the graph of f lies (strictly) above the chord joining $(s, f(s))$ and $(t, f(t))$. That is,

$$f(x) > f(s) + \left(\frac{f(t) - f(s)}{t - s}\right)(x - s), \quad s < x < t,$$

for then we just integrate both sides, as before, concluding that $(t-s)^{-1} \int_s^t f(x)\,dx > [f(t) + f(s)]/2$.

Now if f is not convex, then, for some pair of points $a < s < t < b$ and some point

$$x = \left(\frac{t - x}{t - s}\right)s + \left(\frac{x - s}{t - s}\right)t$$

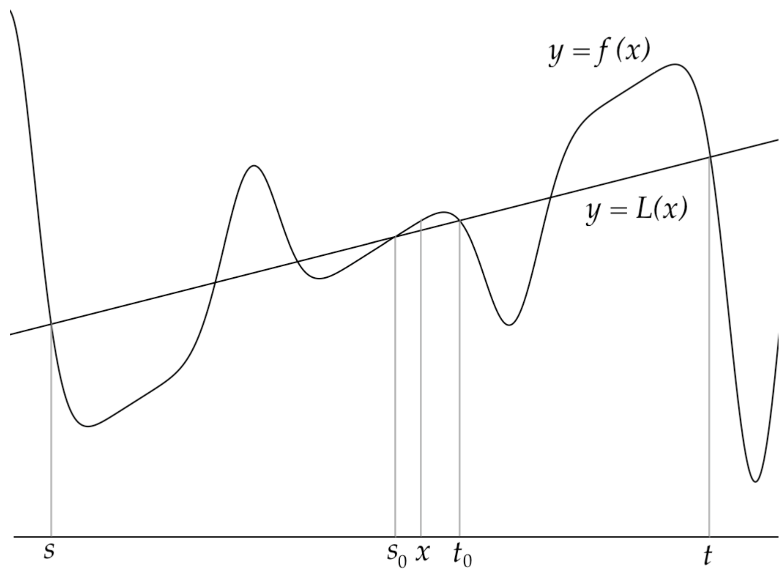
we have

$$f(x) > \left(\frac{t - x}{t - s}\right)f(s) + \left(\frac{x - s}{t - s}\right)f(t) = f(s) + \left(\frac{f(t) - f(s)}{t - s}\right)(x - s) = L(x).$$

As suggested by Mr. Ghosh in class, we next consider:

$$s_0 = \sup\{s' : s \le s' \le x,\ f(s') = L(s')\} \quad \text{and} \quad t_0 = \inf\{t' : x \le t' \le t,\ f(t') = L(t')\}.$$

Each of the sets above is closed and bounded, so s_0 and t_0 exist and satisfy $s_0 < x < t_0$ and $f(s_0) = L(s_0)$, $f(t_0) = L(t_0)$. (The continuity of both f and L has come into play here.) Moreover, $f(y) > L(y)$ for all $s_0 < y < t_0$. (We can't have $f(y) = L(y)$, and the intermediate value theorem tells us that we can't have $f(y) < L(y)$.) All that remains is to note that the line through $(s_0, f(s_0))$ and $(t_0, f(t_0))$ coincides with the line through $(s, f(s))$ and $(t, f(t))$; that is, the chord joining $(s_0, f(s_0))$ and $(t_0, f(t_0))$ lies on the chord joining $(s, f(s))$ and $(t, f(t))$. (See the figure on the back.) $\square$



What makes this problem especially interesting is that there's a companion result (see Theorem **125** in Hardy, Littlewood, and Polya).

10′. Let $f : (a, b) \to \mathbb{R}$ be continuous. Then f is convex if and only if, for all $a < s < t < b$, we have $(t-s)^{-1} \int_s^t f(x)\, dx \geq f((s+t)/2)$.

The forward implication is easy: If f is convex and if $a < s < t < b$, we can consider a supporting line at $(s+t)/2$; that is, we can find an m such that

$$f(x) \geq f\left(\frac{s+t}{2}\right) + m\left(x - \left(\frac{s+t}{2}\right)\right)$$

for all x. Integration over $[\,s, t\,]$ then yields:

$$\int_s^t f(x)\, dx \geq (t-s) f\left(\frac{s+t}{2}\right)$$

because

$$\int_s^t \left(x - \left(\frac{s+t}{2}\right)\right) dx = 0.$$

I'm not sure how to prove the backward implication. Anyone interested?

Elementary Means

Before we pursue more inequalities, let's introduce some notation. Given a vector $x = (x_i)$ in $\mathbb{R}^n$ with *positive* entries and a real number $p \neq 0$, we write

$$S_p(x) = \left(\sum_{i=1}^n x_i^p \right)^{1/p} \qquad \text{and} \qquad M_p(x) = \left(\frac{1}{n} \sum_{i=1}^n x_i^p \right)^{1/p}.$$

The expression $S_p(x)$ is sometimes written as $\|x\|_p$. The expression $M_p(x)$ is called the (simple) *mean of* of x *of order* p. Of course, we're somewhat familiar with such expression when $p > 0$, but we will have occasion to explore the full range of values—even $p = \pm\infty$!

Given a set of *positive* weights $\alpha_1, \ldots, \alpha_n > 0$ with $\alpha_1 + \cdots + \alpha_n = 1$, we write

$$M_p(x, \alpha) = \left(\sum_{i=1}^n \alpha_i x_i^p \right)^{1/p}.$$

$M_p(x, \alpha)$ is called the *weighted mean of* x *of order* p.

Please note that all of these expression are *positive homogeneous*; for example,

$$M_p(kx, \alpha) = k M_p(x, \alpha) \quad \text{for} \quad k > 0,$$

where $kx = (kx_i)$. They're also *increasing* functions of x in the sense that $M_p(x, \alpha) \leq M_p(y, \alpha)$ whenever $x_i \leq y_i$ for all i.

$M_1(x)$ is called the *arithmetic mean* of x; $M_2(x)$ is sometimes called the *root-mean-square* of x; $M_{-1}(x)$ is called the *harmonic mean* of x. One of our goals in this section is to deduce suitable expressions for M_∞, $M_{-\infty}$, and M_0. First, a few simple

Observations.

1. $\min\{ x_1, \ldots, x_n \} \leq M_p(x, \alpha) \leq \max\{ x_1, \ldots, x_n \}$. Equality occurs (in either inequality) only if $x_1 = \cdots = x_n$.

This is reasonably clear if $p > 0$. If $p < 0$, simplest might be to appeal to the fact that

$$M_{-p}(x, \alpha) = \frac{1}{M_p(1/x, \alpha)}$$

where $1/x = (1/x_i)$.

2. $M_0(x,\alpha) \equiv \lim_{p\to 0} M_p(x,\alpha) = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$. Thus, in particular, we also have $\min\{ x_1, \ldots, x_n \} \le M_0(x,\alpha) \le \max\{ x_1, \ldots, x_n \}$.

Proof. First write

$$M_p(x,\alpha) = \exp\left\{ \frac{1}{p} \log\left(\sum_{i=1}^n \alpha_i x_i^p \right) \right\}.$$

Next, we appeal to l'Hôpital's rule to find

$$\begin{aligned}
\lim_{p\to 0} \frac{\log\left(\sum_{i=1}^n \alpha_i x_i^p\right)}{p} &= \lim_{p\to 0} \frac{\sum_{i=1}^n \alpha_i x_i^p \log x_i}{\sum_{i=1}^n \alpha_i x_i^p} \\
&= \sum_{i=1}^n \alpha_i \log x_i \\
&= \log\left(x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n} \right). \quad \square
\end{aligned}$$

3. $M_\infty(x) \equiv \lim_{p\to+\infty} M_p(x,\alpha) = \max\{ x_1, \ldots, x_n \}$ and $M_{-\infty}(x) \equiv \lim_{p\to-\infty} M_p(x,\alpha) = \min\{ x_1, \ldots, x_n \}$.

Proof. Suppose that $x_k = \max\{ x_1, \ldots, x_n \}$. Then $\alpha_k^{1/p} x_k \le M_p(x,\alpha) \le x_k$ and $\alpha_k^{1/p} \to 1$ as $p \to +\infty$. The other case is similar but, again, we could just appeal to the fact that $M_{-p}(x,\alpha) = [\, M_p(1/x,\alpha) \,]^{-1}$. $\square$

4. For $s < t$, we have $M_s(x,\alpha) \le M_t(x,\alpha)$ with equality only if $x_1 = \cdots = x_n$.

Proof. First consider the case $s = 1$. Given $t > 1$, let t' be the conjugate exponent: $\frac{1}{t} + \frac{1}{t'} = 1$. Then, from Hölder's inequality,

$$\sum_{i=1}^n \alpha_i x_i = \sum_{i=1}^n \alpha_i^{1/t} x_i \alpha_i^{1/t'} \le \left(\sum_{i=1}^n \alpha_i x_i^t \right)^{1/t}$$

because $\sum_{i=1}^n \alpha_i = 1$. That is, $M_1(x,\alpha) \le M_t(x,\alpha)$. Equality can only occur if $(\alpha_i x_i^t)$ is proportional to (α_i); i.e., if and only if (x_i^t) is proportional to $(1, 1, \ldots)$; i.e., if and only if x is *constant.*

Now consider the case $0 < s < t$. In this case, $t/s > 1$ and, hence,

$$\left(\sum_{i=1}^n \alpha_i x_i^s \right)^{1/s} \le \left(\sum_{i=1}^n \alpha_i x_i^{s\cdot(t/s)} \right)^{(s/t)(1/s)} = \left(\sum_{i=1}^n \alpha_i x_i^t \right)^{1/t}.$$

The case $s = 0$ is very similar. Given $t > 0$,

$$\left(x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}\right)^t = (x_1^t)^{\alpha_1} (x_2^t)^{\alpha_2} \cdots (x_n^t)^{\alpha_n} \le \sum_{i=1}^n \alpha_i x_i^t$$

by the AGM. That is, $M_0(x, \alpha) \le M_t(x, \alpha)$.

The remaining cases follow easily from what we've already shown. For example, $M_0(x, \alpha) \le M_1(x, \alpha)$, for all x, implies that $M_0(1/x, \alpha) \le M_1(1/x, \alpha)$; that is,

$$x_1^{-\alpha_1} x_2^{-\alpha_2} \cdots x_n^{-\alpha_n} \le \sum_{i=1}^n \alpha_i x_i^{-1} \quad \Longrightarrow \quad x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n} \ge \left(\sum_{i=1}^n \alpha_i x_i^{-1}\right)^{-1}.$$

In other words, $M_0(x, \alpha) \ge M_{-1}(x, \alpha)$.

Of particular interest are these: $M_{-\infty} \le M_{-1} \le M_0 \le M_1 \le M_2 \le M_\infty$. $\square$

Minkowski's Inequality. *Let x, y, $\alpha \in \mathbb{R}^n$ with positive entries and with $\alpha_1 + \cdots + \alpha_n = 1$. Then*

(i) $M_p(x + y, \alpha) \le M_p(x, \alpha) + M_p(y, \alpha)$ *for* $1 \le p \le \infty$;

(ii) $M_p(x + y, \alpha) \ge M_p(x, \alpha) + M_p(y, \alpha)$ *for* $-\infty \le p \le 1$.

If $p \ne 1$ is finite, then equality can only occur if x and y are proportional.

Proof. Of course, equality always occurs (in both (i) and (ii)) when $p = 1$, so we may suppose that $p \ne 1$.

First suppose that $1 < p < \infty$ and let p' satisfy $1/p + 1/p' = 1$. Then $(p - 1)p' = p$. Next,

$$\begin{aligned} M_p(x + y, \alpha)^p &= \sum_{i=1}^n \alpha_i (x_i + y_i)^p = \sum_{i=1}^n \alpha_i (x_i + y_i)(x_i + y_i)^{p-1} \\ &= \sum_{i=1}^n \alpha_i x_i (x_i + y_i)^{p-1} + \sum_{i=1}^n \alpha_i y_i (x_i + y_i)^{p-1} \\ &\le \left[\left(\sum_{i=1}^n \alpha_i x_i^p\right)^{1/p} + \left(\sum_{i=1}^n \alpha_i y_i^p\right)^{1/p}\right] \left(\sum_{i=1}^n \alpha_i (x_i + y_i)^{(p-1)p'}\right)^{1/p'} \qquad (1) \\ &= \Big[M_p(x, \alpha) + M_p(y, \alpha)\Big] \cdot M_p(x + y, \alpha)^{p-1}. \end{aligned}$$

Thus, $M_p(x+y,\alpha) \le M_p(x,\alpha) + M_p(y,\alpha)$. Equality occurs only if equality occurs in *both* applications of Hölder's inequality (in (1)); that is, only if (x_i^p) is proportional to $((x_i+y_i)^p)$ is proportional to (y_i^p); all of which translates to: x is proportional to y.

For $0 < p < 1$, we know that Hölder's inequality *reverses*, so (nearly) the same proof yields $M_p(x+y,\alpha) \ge M_p(x,\alpha) + M_p(y,\alpha)$. The case $-\infty < p < 0$ follow for precisely the same reason for, in this case, the conjugate exponent p' satisfies $0 < p' < 1$ (and Hölder's inequality reverses in this case, too).

The three remaining cases are relatively easy to handle. We've actually shown the case $p = 0$, but using different notation:

$$\begin{aligned}
\frac{M_0(x,\alpha) + M_0(y,\alpha)}{M_0(x+y,\alpha)} &= \frac{x_1^{\alpha_1}\cdots x_n^{\alpha_n} \;+\; y_1^{\alpha_1}\cdots y_n^{\alpha_n}}{(x_1+y_1)^{\alpha_1}\cdots(x_n+y_n)^{\alpha_n}} \\
&= \left(\frac{x_1}{x_1+y_1}\right)^{\alpha_1}\cdots\left(\frac{x_n}{x_n+y_n}\right)^{\alpha_n} + \left(\frac{y_1}{x_1+y_1}\right)^{\alpha_1}\cdots\left(\frac{y_n}{x_n+y_n}\right)^{\alpha_n} \\
&\le \alpha_1\,\frac{x_1}{x_1+y_1} + \cdots + \alpha_n\,\frac{x_n}{x_n+y_n} \;+\; \alpha_1\,\frac{y_1}{x_1+y_1} + \cdots + \alpha_n\,\frac{y_n}{x_n+y_n} \\
&= 1!
\end{aligned}$$

Equality can only occur if each of the sequences $(x_i/(x_i+y_i))$ and $(y_i/(x_i+y_i))$ are constant, from which you'll quickly deduce that x and y must be proportional.

Minkowski's inequality also holds in the cases $p = \pm\infty$, but the case for equality is a bit different. For example,

$$M_\infty(x+y) = \max\{x_1+y_1,\ldots,x_n+y_n\} \le M_\infty(x) + M_\infty(y).$$

Equality can only occur if x and y attain their maximum values at the same coordinate; i.e., if and only if, for some k, we have $x_k = M_\infty(x)$ and $y_k = M_\infty(y)$. $\square$

The Passage to Infinite Series

It's natural to ask whether our work on finite sums (and elementary means) extends to infinite sums. For the most part, everything we've done has a satisfactory analogue in the

infinite series case, however the proofs can be surprisingly difficult—simply passing to a limit is often not enough.

For now, we'll settle for stating a few of these extensions (with the occasional proof). As we'll see, there are more convenient paths to all of this (and more) through *integrals*.

Given a sequence of *positive* weights $\alpha = (\alpha_i)$ satisfying $\sum_{i=1}^{\infty} \alpha_i = 1$ and a *positive* number $0 < p < \infty$, we define

$$M_p(x, \alpha) = \left(\sum_{i=1}^{\infty} \alpha_i x_i^p \right)^{1/p}$$

for $x = (x_i)$ *nonnegative*. We'll forego the case $p < 0$, but we will consider

$$M_0(x, \alpha) = \prod_{i=1}^{\infty} x_i^{\alpha_i} = \exp\left(\sum_{i=1}^{\infty} \alpha_i \log x_i \right)$$

and

$$M_\infty(x) = \sup_{i \ge 1} x_i = \text{l.u.b.}\{\, x_i : i \ge 1 \,\}.$$

Facts.

1. If M_s is finite for some $0 < s < \infty$, then M_r is finite for all $0 < r < s$ and, in fact, $M_r \le M_s$ with equality if and only if x is constant. Moreover, M_0 is finite (or zero) in this case (i.e., M_0 does not diverge to $+\infty$) and, in fact, $M_0 = \lim_{r \to 0^+} M_r$.

2. $M_0 \le M_1$ (AGM) with equality if and only if x is constant.

Proof. From the inequality $\log x \le x - 1$ we have $\log x_k - \log M_1 \le (x_k/M_1) - 1$. Thus,

$$\sum_{k=1}^{\infty} \alpha_k \Big(\log x_k - \log M_1 \Big) \;\le\; \sum_{k=1}^{\infty} \alpha_k \left(\frac{x_k}{M_1} - 1 \right) \tag{2}$$

or, $\log M_0 - \log M_1 \le 1 - 1 = 0$. Equality here would force equality in (2), which would mean that $x_k = M_1$ for all k. □

3. If $x = (x_k)$ is *bounded*, then $\lim_{r \to +\infty} M_r = M_\infty$.

Virtually all of our main results hold in the infinite series case with, at worst, minor modifications.

The Passage to Integrals

In what follows, we will consider *nonnegative*, *finite* (almost everywhere) functions $f : I \to \mathbb{R}$, all defined on some common interval (or measurable set) I, and all assumed to be (Riemann or Lebesgue) *integrable*—meaning that $\int_I f(x)\,dx$ *exists* as a finite real number. In situations where the particular interval has no bearing on the argument or, more commonly, is the same for all integrals, we may suppress it (and x) and simply write $\int f$.

We will only pursue a couple of (very familiar) inequalities.

Hölder's Inequality. *Let f, $g : I \to \mathbb{R}$ be nonnegative integrable functions and let λ, $\mu \geq 0$ with $\lambda + \mu = 1$. Then $f^\lambda g^\mu$ is integrable and satisfies*

$$\int f^\lambda g^\mu \;\leq\; \left(\int f\right)^\lambda \left(\int g\right)^\mu .$$

Equality can only occur if, for some constants A and B, not both zero, we have $Af = Bg$.

Proof. The proof is quite familiar by now: We may suppose that $u = \int f \neq=$ and $v = \int g \neq 0$, in which case we have

$$\left(\frac{f(x)}{u}\right)^\lambda \left(\frac{g(x)}{v}\right)^\mu \;\leq\; \lambda\,\frac{f(x)}{u} + \mu\,\frac{g(x)}{v},$$

with equality (everywhere) if and only if $f/u = g/v$. Now we integrate both sides:

$$\frac{1}{u^\lambda v^\mu}\int f^\lambda g^\mu \;\leq\; \frac{\lambda}{u}\int f + \frac{\mu}{v}\int g \;=\; 1. \qquad \square$$

A particular case is worth repeating:

Corollary. *Let $1 < p < \infty$ and let $\frac{1}{p} + \frac{1}{q} = 1$. If f^p and g^q are integrable, then fg is integrable and satisfies*

$$\int fg \;\leq\; \left(\int f^p\right)^{1/p} \left(\int g^q\right)^{1/q} .$$

Equality can only occur if $Af^p = Bg^q$ for some constants A and B, not both zero.

As you can imagine, if $p < 1$, $p \neq 0$, and if $\int g^q$ is nonzero and finite, then the inequality *reverses*. We will forego the details.

Minkowski is next, but a slightly more general version that will come in handy later.

Minkowski's Inequality. *Let $1 \le p < \infty$. If $|f|^p$ and $|g|^p$ are integrable, then so is $|f+g|^p$ and*

$$\left(\int |f+g|^p\right)^{1/p} \le \left(\int |f|^p\right)^{1/p} + \left(\int |g|^p\right)^{1/p}. \tag{3}$$

If $p=1$, equality can only occur if $fg \ge 0$. If $p>1$, equality can only occur if $fg \ge 0$ and $Af = Bg$ for some nonnegative *constants A and B, not both zero.*

Proof. If $p = 1$, then

$$\int |f+g| \le \int \Big(|f| + |g|\Big) = \int |f| + \int |g|, \tag{4}$$

hence $|f+g|$ is integrable. Equality in (4) forces $|f+g| = |f| + |g|$ (everywhere) and, hence, we must have $fg \ge 0$ (that is, $f(x)$ and $g(x)$ must have the same sign for all x).

If $p > 1$, first note that

$$\begin{aligned} |f+g|^p &\le \Big[\,|f| + |g|\,\Big]^p \\ &\le \Big[\,2\max\{\,|f|, |g|\,\}\Big]^p \\ &= 2^p \max\{\,|f|^p, |g|^p\,\} \\ &\le 2^p\Big(|f|^p + |g|^p\Big). \end{aligned}$$

Thus, $|f+g|^p$ will be integrable. Now let $q = p/(p-1)$ and apply Hölder's inequality:

$$\begin{aligned} \int |f+g|^p &= \int |f+g| \cdot |f+g|^{p-1} \\ &\le \int |f| \cdot |f+g|^{p-1} + \int |g| \cdot |f+g|^{p-1} && (5) \\ &\le \left\{\left(\int |f|^p\right)^{1/p} + \left(\int |g|^p\right)^{1/p}\right\}\left(\int |f+g|^p\right)^{1-(1/p)}. && (6) \end{aligned}$$

And (3) follows. Note that equality in (3) forces equality in both (5) and (6). Thus we have $fg \ge 0$ and

$$A|f|^p \sim B|g|^p \sim C|f+g|^p$$

(where "$\sim$" means "is proportional to"), which forces $A'f = B'g$ for some *nonnegative* constants A' and B', not both zero. $\square$

We next, all too briefly, consider integral means. Given a positive, integrable weight function $\alpha(x)$, we define

$$M_p(f,\alpha) \;=\; \left(\int_I f(x)^p\, \alpha(x)\, dx\right)^{1/p}$$

for $0 < p < \infty$ and $f \geq 0$ (and measurable, say, or even continuous). We also define

$$M_0(f,\alpha) \;=\; \exp\left(\int_I \alpha(x)\, \log f(x)\, dx\right)$$

and

$$M_\infty(f) \;=\; \sup_{x\in I} f(x) \quad (=\; \operatorname{ess.sup}_{x\in I} f(x)).$$

We could then develop the theory of the means $M_p(f,\alpha)$ in complete analogy to our previous cases. We will, however, settle for a single observation:

If $M_s(f,\alpha) < \infty$ for some $0 < s < \infty$, then $M_r(f,\alpha) < \infty$ for all $0 < r < s$ and $M_r(f,\alpha) \leq M_s(f,\alpha)$, with equality if and only if f is constant.

Proof. As before, the proof follows from Hölder's inequality, applied to $p = s/r > 1$.

$$\begin{aligned}\left(\int f^r\,\alpha\right)^{1/r} &= \left(\int f^r\,\alpha^{r/s}\,\alpha^{1-(r/s)}\right)^{1/r}\\ &\leq \left(\int f^s\,\alpha\right)^{(r/s)(1/r)} \left(\int \alpha\right)^{(1/r)-(1/s)}\\ &= \left(\int f^s\,\alpha\right)^{1/s}. \quad \square\end{aligned}$$

This result should be compared with the non-weighted setting, the so-called L_p-norms

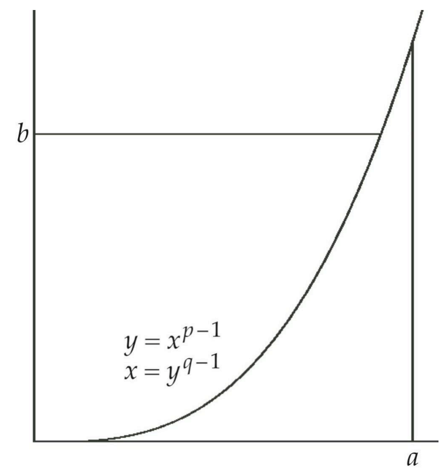
$$\|f\|_p = \left(\int_I f(x)^p\, dx\right)^{1/p} \qquad (0 < p < \infty),$$

which are incomparable if I has infinite length and which otherwise satisfy

$$\|f\|_r \leq (b-a)^{(1/r)-(1/s)} \|f\|_s$$

if $r < s$ and $I = [\,a, b\,]$ (to see this, just set $\alpha \equiv 1$ in our previous calculation).

A few calculus problems



1. Let $1 < p < \infty$ and let q satisfy $1/p + 1/q = 1$; in other words, $(p-1)(q-1) = 1$. By examining the graph of $y = x^{p-1}$ (a.k.a., $x = y^{q-1}$), pictured at right, argue that

$$ab \le \int_0^a x^{p-1} dx + \int_0^b y^{q-1} dy,$$

with equality if and only if $b = a^{p-1}$ (equivalently, $a^p = b^q$). More generally, if $y = f(x)$ and $x = g(y)$ are strictly increasing, continuous, inverses of one another for $x, y > 0$, argue that

$$ab \le \int_0^a f(x)\, dx + \int_0^b g(y)\, dy,$$

with equality if and only if $b = f(a)$.

2. Establish the following (using, for example, the mean value theorem).

(a) $x/(1+x) \le \log(1+x) \le x$ for $x > -1$, with equality (in either inequality) only at $x = 0$.

(b) $e^x \ge 1 + x$ with equality only at $x = 0$. Conclude that $e^x > (1 + (x/n))^n$.

(c) Deduce that $\log(1+x) \le x/(1-x)$ for $-1 < x < 1$, with equality only at $x = 0$.

(d) Deduce that $e^x \le 1/(1-x)$ for $x < 1$, with equality only at $x = 0$.

3. Establish the following variations on Bernoulli's inequality.

(i) $(1+x)^\alpha \le 1 + \alpha x$ for $x \ge -1$ and $0 < \alpha < 1$.

(ii) $(1+x)^\alpha \ge 1 + \alpha x$ for $x \ge -1$ and either $\alpha < 0$ or $\alpha > 1$.

(iii) $(1-x)^\alpha \le 1 - \alpha x$ for $0 \le x \le 1$ and $0 < \alpha < 1$. [Hint: Write $1/(1-x) = 1 + x/(1-x)$.]

(iv) $(1-x)^\alpha \ge 1 - \alpha x$ for $0 \le x \le 1$ and $\alpha > 1$. [Hint: If $0 < x < 1/\alpha$, consider $(1 - \alpha x)^{1/\alpha}$.]

4. Which is bigger, π^e or e^π?

A few algebra problems

5. Show that $\sqrt[n+1]{ab^n} \le \dfrac{a+nb}{n+1}$ for all $n = 1, 2, \ldots$, with equality if and only if $a = b$.

6. (i) Show that $\dfrac{1}{x-1} + \dfrac{1}{x} + \dfrac{1}{x+1} > \dfrac{3}{x}$ for $x > 1$.

(ii) Use (i) to show that the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

7. Without appealing to calculus, establish the following, where a, b, $c > 0$.

(i) $ab + bc + ca \le a^2 + b^2 + c^2$

(ii) $9abc \le (a+b+c)(ab+bc+ca)$

(iii) $abc(a+b+c) \le a^2b^2 + b^2c^2 + c^2a^2$

8. Show that $64 \le \left(1 + \dfrac{1}{x}\right)\left(1 + \dfrac{1}{y}\right)\left(1 + \dfrac{1}{z}\right)$ for x, y, $z > 0$, $x + y + z = 1$.

9. Show that $x + 1/(xy) + y^2 \ge 5(2^{-4/5})$ for any x, $y > 0$. Find values of x and y that yield equality.

10. Show that the rectangular box of volume V having minimum surface area S is a cube. [Hint: Show that $V^{2/3} \le S/6$, with equality occurring if and only if all three edges have equal length.]

11. Show that the right circular cylinder of volume V having least surface area S has its diameter equal to its height.

12. Given $a_1, \ldots, a_n > 0$, show that $\left(\sum_{i=1}^{n} a_i\right)\left(\sum_{i=1}^{n} \dfrac{1}{a_i}\right) \ge n^2$, with equality if and only if $a_1 = \cdots = a_n$.

6. Show that $64 \le \left(1+\dfrac{1}{x}\right)\left(1+\dfrac{1}{y}\right)\left(1+\dfrac{1}{z}\right)$ for $x,\ y,\ z > 0$, $x+y+z=1$.

Solution. The function $f(x) = \log\left(1+\frac{1}{x}\right) = \log(1+x) - \log x$ is strictly convex for $x > 0$ because $f''(x) = \frac{1}{x^2} - \frac{1}{(1+x)^2} > 0$ for $x > 0$. Thus, $f(x)+f(y)+f(z) \ge 3f\left(\frac{x+y+z}{3}\right) = 3f(\frac{1}{3}) = \log(4^3)$. Exponentiating, this becomes:

$$\left(1+\frac{1}{x}\right)\left(1+\frac{1}{y}\right)\left(1+\frac{1}{z}\right) \ge 64.$$

By strict convexity, equality can only occur if $x = y = z = 1/3$.

Notation: Fix $n \in \mathbb{N}$ and $\alpha = (\alpha_1, \ldots, \alpha_n)$, where $\alpha_i > 0$ for all i and $\alpha_1 + \cdots + \alpha_n = 1$. For each nonzero real number t and each $x = (x_1, \ldots, x_n)$ with $x_i > 0$, $i = 1, \ldots, n$, we define the weighted mean $M_t(x, \alpha)$ of order t by $M_t(x, \alpha) = (\alpha_1 x_1^t + \cdots \alpha_n x_n^t)^{1/t}$. In the remaining cases, $t = 0, \pm\infty$, we define $M_0(x, \alpha) = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, $M_\infty(x, \alpha) = M_\infty(x) = \max\{x_1, \ldots, x_n\}$, and $M_{-\infty}(x, \alpha) = M_{-\infty}(x) = \min\{x_1, \ldots, x_n\}$, respectively. In the case $\alpha_1 = \cdots = \alpha_n = 1/n$, we often omit reference to α and write $M_t(x)$, which is called the simple mean of order t.

1. Prove Hölder's inequality in the case $t = 1$ ($t' = \pm\infty$) by showing $M_1(x)M_{-\infty}(y) \leq M_1(xy) \leq M_1(x)M_\infty(y)$, where xy denotes the sequence $(x_1y_1, \ldots, x_ny_n)$. When does equality occur (in either of these inequalities)?
2. Prove Liapounov's inequality: If $0 < r < s < t$ and if we write $s = \lambda r + \mu t$, where λ, $\mu > 0$, $\lambda + \mu = 1$, then $M_s(x)^s \leq M_r(x)^{r\lambda} M_t(x)^{t\mu}$. Equality can only occur if all of the x_i are equal. Upon taking logarithms, Liapounov's inequality tells us that the function $s \mapsto s \log M_s$ is convex (for fixed x and α).

Recall that we have also defined the sums $S_t(x) = (x_1^t + \cdots + x_n^t)^{1/t}$ for $t > 0$. Note that $S_t(x) = n^{1/t} M_t(x)$. A somewhat more common notation is $\|x\|_t = (x_1^t + \cdots + x_n^t)^{1/t}$. These expressions are rarely used for $t < 0$, thus we are free to consider nonnegative x_i.

3. If $0 < p < q$, show that $S_q(x) \leq S_p(x)$. Equality can only occur if all but one of the x_i is zero. [Hint: The inequality is homogenous in x, thus we may assume that $S_p(x) = 1$.] This is often called Jensen's inequality. Please compare this result with the fact that $M_p(x) \leq M_q(x)$.
4. Show that $\lim_{t\to\infty} S_t(x) = M_\infty(x)$. Thus, we define $S_\infty(x) = M_\infty(x)$.
5. Given $x > 0$, show that $\lim_{t\to 0^+} (1 + x^t)^{1/t} = +\infty$. Conclude that $\lim_{t\to 0^+} S_t(x) = +\infty$ if x has two or more nonzero coordinates.
6. Show that the sums S_t satisfy Hölder's inequality: If $p > 1$ and q satisfies $1/p + 1/q = 1$, then $S_1(xy) \leq S_p(x)S_q(y)$. If $0 < p < 1$, the inequality reverses: $S_1(xy) \geq S_p(x)S_q(y)$. In every case, equality can only occur if x and y are proportional.
7. Show that $S_1(xy) \leq S_1(x)S_\infty(y)$ (this is Hölder's inequality in the case $p = 1$, $q = \infty$). When does equality occur?
8. Show that if p, $q > r > 0$ satisfy $1/p + 1/q = 1/r$, then $S_r(xy) \leq S_p(x)S_q(y)$.
9. Given $0 < p < r < q$, write $r = \lambda p + \mu q$, where λ, $\mu > 0$, $\lambda + \mu = 1$. Show that $S_r(x)^r \leq S_p(x)^{p\lambda} S_q(x)^{q\mu}$. Equality can only occur if all of the nonzero x_i are equal.

10. Show that the sums S_t satisfy Minkowski's inequality: For $t > 1$ we have $S_t(x+y) \leq S_t(x) + S_t(y)$, where $x+y$ denotes the sequence $(x_1+y_1, \ldots, x_n+y_n)$. For $0 < t < 1$, we have $S_t(x+y) \geq S_t(x) + S_t(y)$. In every case, equality can only occur if x and y are proportional. (Of course, equality always holds if $t = 1$.)

11. Show that $S_\infty(x+y) \leq S_\infty(x) + S_\infty(y)$. When does equality occur in this case?

For $1 \leq p \leq \infty$, the expression $\|x\|_p = S_p(x)$ defines a *norm* on $\mathbb{R}^n$. In other words, $\|x\|_p$ satisfies: (i) $\|x\|_p \geq 0$ for all $x \in \mathbb{R}^n$ and $\|x\|_p = 0$ if and only if $x = 0$; (ii) $\|ax\|_p = |a|\,\|x\|_p$ for any scalar $a \in \mathbb{R}$; and (iii) $\|x+y\|_p \leq \|x\|_p + \|y\|_p$ for any x, $y \in \mathbb{R}^n$ (which is typically called the *triangle inequality* in this context). For $0 < p < 1$, (iii) reverses, thus $\|x\|_p$ is not a norm in this case. However, as we'll see directly, for $0 < p < 1$, the expression $d(x,y) = \|x-y\|_p^p$ defines a *translation invariant metric* on $\mathbb{R}^n$.

12. If $p \geq 1$, show that $\sum_{i=1}^n (x_i+y_i)^p \geq \sum_{i=1}^n x_i^p + \sum_{i=1}^n y_i^p$. If $0 < p \leq 1$, the inequality reverses. If $p \neq 1$, equality can only occur if $x_i y_i = 0$ for all i (in other words, x_i and y_i cannot both be nonzero).

13. For $0 < p < 1$, show that the expression $d(x,y) = \|x-y\|_p^p$ satisfies: (i) $d(x,y) \geq 0$ for all x, $y \in \mathbb{R}^n$ and $d(x,y) = 0$ if and only if $x = y$; (ii) $d(x,y) = d(y,x)$ for all x, $y \in \mathbb{R}^n$; (iii) $d(x,y) = d(x-y,0) = d(x+z,y+z)$ for all x, y, $z \in \mathbb{R}^n$; and (iv) $d(x,y) \leq d(x,z) + d(z,y)$ for all x, y, $z \in \mathbb{R}^n$.

Stirling's Formula

Recall (from Problem 14, Problem Set 1) that $(n/3)^n < n! < (n/2)!$ for $n \geq 6$. Thus, it's reasonable to ask whether $n! \sim (n/a)^n$ for some constant a (where $\sim$ means that the ratio $n!/(n/a)^n \to 1$ as $n \to \infty$). If you guessed that $a = e$, you'd be right—but the precise order of magnitude (including error estimates) takes a bit more work.

We begin with a clever summation formula, due essentially to Euler.

Euler-Maclaurin Summation. *If f has a continuous derivative on $[1, n]$, then*

$$\sum_{k=1}^{n} f(k) \;=\; \int_1^n f(x)\,dx + \int_1^n \big(x - [x]\big)\, f'(x)\,dx + f(1),$$

where $[x]$ denotes the greatest integer $\leq x$.

Proof. (If you happen to know Stieltjes integration, there's a very short proof! For simplicity, we'll settle for a slightly longer but still elementary proof.) We begin with the "error":

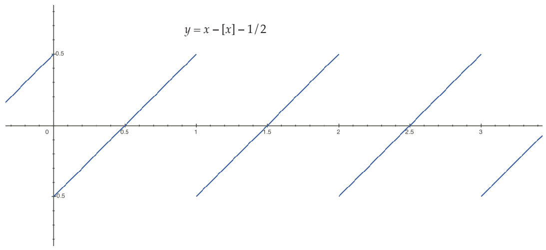
$$\sum_{k=1}^{n-1} f(k) - \int_1^n f(x)\,dx \;=\; \sum_{k=1}^{n-1} \int_k^{k+1} \big(f(k) - f(x)\big)\,dx.$$

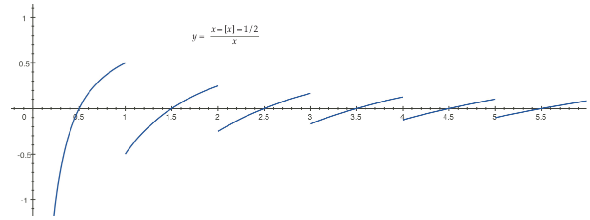
(Please note that the upper limit on the sums is now $n-1$.) We are going to integrate by parts on the right-hand side, using the variables $u = f(k) - f(x)$ and $v = x - k - 1$ (!). With this choice we'll have $u(k) = 0 = v(k+1)$, which will simplify things considerably. Watch closely!

$$\begin{aligned}\sum_{k=1}^{n-1} f(k) - \int_1^n f(x)\,dx \;&=\; \sum_{k=1}^{n-1} \int_k^{k+1} \big(x - k - 1\big)\, f'(x)\,dx \\ &=\; \sum_{k=1}^{n-1} \int_k^{k+1} \big(x - [x] - 1\big)\, f'(x)\,dx \\ &=\; \int_1^n \big(x - [x]\big)\, f'(x)\,dx + f(1) - f(n).\end{aligned}$$

Adding $f(n)$ to both sides completes the proof. $\square$

Now Euler made another helpful observation: If we replace $x - [x]$ by $x - [x] - 1/2$, we'll enhance the likelihood of convergence of the second integral (as $n \to \infty$) because we'll introduce the possibility of *cancellations*.





Corollary. $\displaystyle\sum_{k=1}^{n} f(k) \;=\; \int_1^n f(x)\,dx + \int_1^n \big(\,x - [x] - 1/2\,\big)\, f\,'(x)\,dx + [\,f(1) + f(n)\,]/2.$

Corollary. $\displaystyle\log(\,n!\,) \;=\; (n + 1/2)\log n - n + 1 + \int_1^n \frac{x - [x] - 1/2}{x}\;dx.$

Proof.

$$\begin{aligned}\log(\,n!\,) \;&=\; \sum_{k=1}^{n} \log k \\ &=\; \int_1^n \log x\,dx + \int_1^n \frac{x - [x] - 1/2}{x}\;dx + \frac{1}{2}\log n \\ &=\; (n + 1/2)\log n - n + 1 + \int_1^n \frac{x - [x] - 1/2}{x}\;dx. \qquad \square\end{aligned}$$

In other words, we've just shown that

$$a_n \;\equiv\; \log\left(\frac{n!\,e^n}{n^{n+1/2}}\right) \;=\; 1 + \int_1^n \frac{x - [x] - 1/2}{x}\;dx. \tag{1}$$

We next show that (a_n) converges using a standard technique from advanced calculus.

Dirichlet's Test for Integrals. *If* $\displaystyle\int_1^x f(t)\,dt$ *is bounded for* $x \ge 1$ *and if* $g(x)$ *decreases to zero as* $x \to \infty$, *then* $\displaystyle\int_1^\infty f(x)\,g(x)\,dx$ *exists (as an improper Riemann integral).*

It's not hard to see that

$$\left|\int_1^x \big(\,x - [x] - 1/2\,\big)\,dx\right| = \left|\int_{[x]}^x \big(\,x - [x] - 1/2\,\big)\,dx\right| \le \int_{1/2}^1 \big(\,x - [x] - 1/2\,\big)\,dx = \frac{1}{8}.$$

Thus, $\displaystyle\int_1^\infty \frac{x - [x] - 1/2}{x}\;dx$ exists. This proves that (a_n) converges to a finite limit and, hence, that $b_n = e^{a_n}$ converges to a positive, finite limit C; that is, we've shown that

$$C \;=\; \lim_{n\to\infty} \frac{n!\,e^n}{n^{n+1/2}} \quad \text{exists.}$$

This result is essentially due to DeMoivre in 1730. Finding the precise value of C, together with some quantitative error estimates, will take a bit more work.

The Gamma Function

We begin with a continuous extension of $n!$ initiated by an observation of Euler's:

$$n! \;=\; \int_0^1 (-\log x)^n \, dx \;=\; \int_0^\infty t^{n-1} e^{-t} \, dt.$$

This motivated Legendre to consider the function

$$\Gamma(x) \;=\; \int_0^\infty t^{x-1} e^{-t} \, dt, \quad x > 0.$$

It's not hard to see that $\Gamma(x) < \infty$ for all $x > 0$. Indeed, given $x > 0$, we can find $t_0 = t_0(x)$ sufficiently large so that $t^{x-1}e^{-t} \le e^{-t/2}$ for all $t \ge t_0$. Thus,

$$\Gamma(x) \;\le\; \int_0^{t_0} t^{x-1} \, dt + \int_{t_0}^\infty e^{-t/2} \, dt \;=\; \frac{t_0^x}{x} + 2e^{-t_0/2} < \infty.$$

We next show that $\Gamma(x)$ is continuous—by showing that it's convex!

Theorem. (i) $\Gamma(1) = 1$.

(ii) $\Gamma(x+1) = x\Gamma(x)$ *hence*, $\Gamma(n+1) = n!$

(iii) Γ *is* log-convex*; that is*, $\log \Gamma(x)$ *is convex.*

(iv) Γ *is convex, hence* continuous.

Proof. (i) is clear: $\int_0^\infty e^{-t} \, dt = 1$. (ii) follows from integration by parts:

$$\begin{aligned}
\Gamma(x+1) \;&=\; \int_0^\infty t^{x-1} e^{-t} \, dt \\
&=\; \int_0^\infty t^{x-1} \, d\big(-e^{-t}\big) \\
&=\; t^x\big(-e^{-t}\big)\Big]_{t=0}^{t=\infty} - \int_0^\infty \big(-e^{-t}\big) x t^{x-1} \, dt \\
&=\; x \int_0^\infty t^{x-1} e^{-t} \, dt \;=\; x\Gamma(x).
\end{aligned}$$

To establish (iii), we'll use Hölder's inequality. Given λ, $\mu \geq 0$ with $\lambda + \mu = 1$, we have:

$$\begin{aligned}\Gamma(\lambda x + \mu y) &= \int_0^\infty t^{(\lambda x + \mu y) - 1} e^{-t}\, dt \\ &= \int_0^\infty \left[t^{x-1} e^{-t} \right]^\lambda \left[t^{y-1} e^{-t} \right]^\mu dt \\ &\leq \Gamma(x)^\lambda\, \Gamma(y)^\mu,\end{aligned}$$

and it follows that $\log \Gamma$ is convex. Finally, (iv) is a general principle: Every log-convex function is also convex. This follows from Young's inequality:

$$\Gamma(\lambda x + \mu y) \;\leq\; \Gamma(x)^\lambda\, \Gamma(y)^\mu \;\leq\; \lambda \Gamma(x) + \mu \Gamma(y). \quad \square$$

Corollary. *As $x \to 0^+$ we have $x\Gamma(x) \to 1$ and, hence, $\Gamma(x) \to +\infty$.*

Proof. $x\Gamma(x) = \Gamma(x+1) \to \Gamma(1) = 1$ as $x \to 0^+$. It follows that $\Gamma(x) = \dfrac{x\Gamma(x)}{x} \to +\infty$ as $x \to 0^+$. $\square$

We next prove a remarkable characterization due to Artin in 1964.

Theorem. *Suppose that $f : (0,\infty) \to (0,\infty)$ satisfies* (i) $f(1) = 1$; (ii) $f(x+1) = xf(x)$; *and* (iii) f *is log-convex. Then* $f = \Gamma$.

Proof. Of course, by (i) and (ii), $f(n+1) = n!$ for $n = 0, 1, 2, \ldots$. Next, given $0 < x \leq 1$, we use (ii) and (iii) to estimate $f(n+1+x)$. Watch closely!

$$\begin{aligned}f(n+1+x) &= f(n+1+x-nx-x+nx+x) \\ &= f\big((1-x)(n+1) + x(n+2)\big) \\ &\leq f(n+1)^{1-x}\, f(n+2)^x \\ &= f(n+1)^{1-x} \left[(n+1) f(n+1) \right]^x \\ &= (n+1)^x\, f(n+1) \\ &= (n+1)^x\, n!\end{aligned}$$

Also,

$$\begin{aligned}
n! \;=\; f(n+1) &= f\big(x(n+x)+(1-x)(n+1+x)\big) \\
&\le f(n+x)^x\, f(n+1+x)^{1-x} \\
&= (n+x)^{-x}\, f(n+1+x)^x\, f(n+1+x)^{1-x} \\
&= (n+x)^{-x}\, f(n+1+x).
\end{aligned}$$

That is, $f(n+1+x) \ge (n+x)^x\, n!$.

Now we use (ii) to write these inequalities in terms of $f(x)$.

$$\begin{aligned}
f(n+1+x) &= (n+x)\, f(n+x) \\
&= (n+x)(n-1+x)\, f(n-1+x) \\
&\;\;\vdots \\
&= (n+x)(n-1+x)\cdots x\, f(x).
\end{aligned}$$

Thus,

$$(n+x)^x\, n! \;\le\; (n+x)(n-1+x)\cdots x\, f(x) \;\le\; (n+1)^x\, n!$$

or, after dividing by $n^x\, n!$,

$$\left(1+\frac{x}{n}\right)^x \;\le\; \frac{(n+x)(n-1+x)\cdots x}{n^x\, n!}\cdot f(x) \;\le\; \left(1+\frac{1}{n}\right)^x.$$

For *fixed* x, both extremes tend to 1 as $n \to \infty$, so we've shown that

$$f(x) \;=\; \lim_{n\to\infty} \frac{n^x\, n!}{(n+x)(n-1+x)\cdots x} \tag{2}$$

for $0 < x \le 1$. We next show that (2) holds for $x > 1$ as well.

Given $x > 1$, there is a positive integer m with $0 < x-m \le 1$. Then

$$\begin{aligned}
f(x) &= (x-1)(x-2)\cdots(x-m)\, f(x-m) \\
&= (x-1)(x-2)\cdots(x-m)\, \lim_{n\to\infty} \frac{n^{x-m}\, n!}{(n+x-m)\cdots(x-m)} \\
&= \lim_{n\to\infty} \frac{n^{x-m}\, n!}{(n+x-m)\cdots x}
\end{aligned}$$

$$= \lim_{n\to\infty} \frac{n^x\, n!}{(n+x)\cdots x} \cdot \frac{(n+x)\cdots(n+x+(m-1))}{n^m} \tag{3}$$
$$= \lim_{n\to\infty} \frac{n^x\, n!}{(n+x)\cdots x}$$

because the second factor in (3) consists of m terms each, top and bottom, and m and x are *fixed*; thus, the limit of this factor is 1 as $n \to \infty$. Thus, (2) holds for all $x > 0$. In other words, there can be *only one* function satisfying the hypotheses of the theorem. $\square$

Corollary. $\Gamma(x) \;=\; \lim_{n\to\infty} \dfrac{n^x\, n!}{(n+x)(n-1+x)\cdots x}$.

In particular, $\Gamma(1/2) \;=\; \lim_{n\to\infty} \dfrac{n^{1/2}\, n!}{(n+1/2)(n-1/2)\cdots(1/2)}$. But we can also find this limit by alternate means:

$$\Gamma(1/2) \;=\; \int_0^\infty t^{-1/2} e^{-t}\, dt = \; 2\int_0^\infty e^{-t}\, d\big(t^{1/2}\big) \;=\; 2\int_0^\infty e^{-u^2}\, du = \;\sqrt{\pi}.$$

Return to Stirling's Formula

If you'll recall, we've shown that $b_n = \dfrac{n!\, e^n}{n^{n+1/2}} \to b > 0$. Thus,

$$\frac{b_n^2}{b_{2n}} \;=\; \frac{(n!)^2\, e^{2n}}{n^{2n+1}} \cdot \frac{(2n)^{2n+1/2}}{(2n)!\, e^{2n}} \;\to\; b.$$

But

$$\frac{2^{2n}\, n!}{(2n)!} \;=\; \frac{2^{2n}\, n!}{(2n)(2n-1)(2n-2)\cdots 3\cdot 2\cdot 1}$$
$$= \frac{n!}{n(n-1/2)(n-1)\cdots(3/2)\cdot 1\cdot(1/2)}$$
$$= \frac{1}{(n-1/2)(n-3/2)\cdots(3/2)\cdot(1/2)}.$$

So,

$$\frac{b_n^2}{b_{2n}} \;=\; \frac{n^{1/2}\, n!}{(n+1/2)(n-1/2)\cdots(3/2)\cdot(1/2)} \cdot \frac{n+1/2}{n} \cdot \sqrt{2} \;\to\; \sqrt{2}\,\Gamma(1/2) \;=\; \sqrt{2\pi}.$$

Thus, we've finally arrived at *Stirling's Formula.*

$$\lim_{n\to\infty} \frac{n!\, e^n}{n^{n+1/2}} \;=\; \sqrt{2\pi}$$

or, in the notation from the beginning of this section, $n! \sim \sqrt{2\pi}\cdot n^{n+1/2}\cdot e^{-n}$.

Stirling's Series

Finally, let's talk about error estimates. In addition to the *qualitative* statement that

$$\frac{n!\, e^n}{\sqrt{2\pi}\cdot n^{n+1/2}} \to 1 \quad \text{as} \quad n \to \infty,$$

Stirling provided *quantitative* estimates on

$$r_n \;=\; \log\left(\frac{n!\, e^n}{\sqrt{2\pi}\cdot n^{n+1/2}}\right).$$

He showed that

$$r_n \;\approx\; \frac{A}{n} - \frac{B}{n^3} + \frac{C}{n^5} - \cdots$$

for certain constants A, B, etc., where $\approx$ means that r_n lies between successive partial sums of the (divergent) alternating series (called *Stirling's series*). For example,

$$\frac{A}{n} - \frac{B}{n^3} \;<\; r_n \;<\; \frac{A}{n} \qquad (n = 1, 2, 3, \ldots)$$

for suitable A and B. We'll settle for the modest estimate:

Lemma. $0 < r_n < 1/12n$ *for all* n. *The constant* $1/12$ *cannot be improved.*

Proof. The proof makes repeated use of the fact that if a sequence (or function) *decreases* to 0 as $n \to \infty$ (resp., $x \to \infty$), then it must be positive. Similarly, if a sequence (or function) *increases* to 0, then it must be negative.

In order to show that $r_n > 0$, then, it suffices to show that (r_n) is *decreasing* (because we already know that $r_n \to 0$). Thus we want to show that

$$\begin{aligned} r_{n+1} - r_n \;&=\; \log\left(\frac{(n+1)!\, e^{n+1}}{\sqrt{2\pi}\cdot (n+1)^{n+1+1/2}} \cdot \frac{\sqrt{2\pi}\cdot n^{n+1/2}}{n!\, e^n}\right) \\ &=\; 1 - (n+1/2)\log\left(1 + \frac{1}{n}\right) \;<\; 0 \end{aligned}$$

for all n. For this, it suffices to show that

$$\log\left(1 + \frac{1}{n}\right) - \frac{1}{n+1/2} \;>\; 0$$

for all n. To this end, notice that the function $g(x) = \log(1+1/x) - 1/(x+1/2)$ tends to 0 as $x \to \infty$. If we can show that g is *decreasing*, we'll know that $g(x) > 0$. But

$$g'(x) \;=\; \frac{1}{(x+1/2)^2} - \frac{1}{x(x+1)} \;=\; \frac{1}{x^2+x+1/4} - \frac{1}{x^2+x} < 0$$

for $x > 0$. Thus we've shown that (r_n) decreases to 0 and, hence, that $r_n > 0$ for all n.

We next find a constant $A > 0$ so that $r_n < A/n$ for all n. For this, it suffices to show that $a_n = r_n - (A/n) < 0$ for all n. But $a_n \to 0$, so it suffices to find A such that (a_n) is *increasing*. That is, we want

$$a_{n+1} - a_n \;=\; 1 - (n+1/2)\log\left(1+\frac{1}{n}\right) + A\left(\frac{1}{n} - \frac{1}{n+1}\right) \;>\; 0$$

for all n. This will be the case if

$$g(x) \;=\; \frac{1}{x+1/2} - \log\left(1+\frac{1}{x}\right) + A\left(\frac{\frac{1}{x} - \frac{1}{x+1}}{x+1/2}\right) \;>\; 0$$

for all $x > 0$. Again, we know that $g(x) \to 0$ as $x \to \infty$, so it would suffice to find A so that $g'(x) < 0$ for all $x > 0$. Hang on!

$$\begin{aligned} g(x) \;&=\; \frac{2}{2x+1} - \log\left(\frac{x+1}{x}\right) + \frac{2A}{x(x+1)(2x+1)} \\ \implies g'(x) \;&=\; \frac{x(x+1) - A(12x^2+12x+2)}{x^2(x+1)^2(2x+1)^2}. \end{aligned}$$

Thus it suffices to find A such that

$$A \;>\; \frac{x(x+1)}{12x^2+12x+2} \;=\; h(x), \qquad x > 0. \tag{4}$$

But $h(x) \to 1/12$ as $x \to \infty$ and

$$h'(x) \;=\; \frac{2x+1}{2(6x^2+6x+1)^2} \;>\; 0$$

for $x > 0$. That is, $h(x)$ *strictly increases* to $1/12$ as $x \to \infty$, so $A = 1/12$ is the *smallest possible upper bound* in (4). Because this is such a roundabout proof, let's summarize what we've done: $A = 1/12 \implies g' < 0 \implies g > 0 \implies (a_n)$ increasing $\implies a_n < 0 \implies r_n < 1/12n$. Phew!! $\square$

Conclusion. *For all* $n = 1, 2, 3, \ldots$, *we have* $1 \;<\; \dfrac{n!\, e^n}{\sqrt{2\pi}\cdot n^{n+1/2}} \;<\; e^{1/12n}$.

Problem Set 4

Notation: A positive function $f : I \to (0, \infty)$ defined on an interval I is said to be *log-convex* if $\log f$ is convex. Equivalently, f is log-convex if $f(\lambda x + \mu y) \le \big[f(x)\big]^{\lambda}\big[f(y)\big]^{\mu}$ whenever $x, y \in I$, $\lambda, \mu \ge 0$, $\lambda + \mu = 1$. From Young's inequality, a log-convex function is also convex; thus, the log-convex functions form a subset of the convex functions.

1. (i) If $a, b > 0$, show that $f(x) = a \cdot b^x$ is log-convex on $\mathbb{R}$.

(ii) Show that $g(x) = x$ is convex but not log-convex on $(0, \infty)$.

2. If f and g are log-convex, show that fg and $f + g$ are log-convex.

3. For x and α fixed, show that the function $f(t) = M_t(x, \alpha)^t$ is log-convex for $t > 0$.

4. Use the log-convexity of $\Gamma(x)$ to show that $\sqrt{\dfrac{2}{n+1}} \le \dfrac{\Gamma(\frac{1}{2}(n+1))}{\Gamma(\frac{1}{2}(n+2))} \le \sqrt{\dfrac{2}{n}}$ for $n = 1, 2, 3, \ldots$.

5. Evaluate $\displaystyle\int_0^\infty t^{1/2} e^{-t^3}\, dt$.

6. Show that $\Gamma\left(n + \dfrac{1}{2}\right) = \dfrac{(2n)!\,\sqrt{\pi}}{n!\,4^n}$ for $n = 0, 1, 2, \ldots$.

7. Show that $\dbinom{2n}{n} \sim \dfrac{2^{2n}}{\sqrt{n\pi}}$. [Recall that $a_n \sim b_n$ means that $a_n/b_n \to 1$.]

8. Establish *Wallis's formula*: $\dfrac{\pi}{2} = \lim\limits_{n\to\infty} \dfrac{2\cdot 2}{1\cdot 3} \cdot \dfrac{4\cdot 4}{3\cdot 5} \cdots \dfrac{(2n)(2n)}{(2n-1)(2n+1)}$.

[Hint: $\pi/2 = (1/2)(\Gamma(1/2))^2$.]

9. Prove *Legendre's duplication formula*: $\Gamma\left(\dfrac{x}{2}\right)\Gamma\left(\dfrac{x+1}{2}\right) = \dfrac{\sqrt{\pi}}{2^{x-1}}\,\Gamma(x)$.

[Hint: Show that $f(x) = (2^{x-1}/\sqrt{\pi})\Gamma(x/2)\Gamma((x+1)/2)$ satisfies the hypotheses of Artin's theorem.]

10. Consider $f(x) = \displaystyle\int_0^\infty \frac{t^{x-1}}{1+t}\, dt$ for $0 < x < 1$. [Euler showed that $f(x) = \pi/\sin(\pi x)$.]

(a) Use the fact that $(1+t)^{-1} = \int_0^\infty e^{-(1+t)s}\, ds$ to show that $f(x) = \Gamma(x)\Gamma(1-x)$.

(b) Show that f is log-convex on $(0, 1)$.

(c) Conclude that $\pi = f(1/2) \le f(x)$ for all $0 < x < 1$.

Inner Product Spaces

An *inner product space* is a vector space X over $\mathbb{C}$ (or $\mathbb{R}$) together with a scalar-valued function $\langle\, x, y\,\rangle$ of two variables, called an *inner product*, satisfying:

(i) $\langle \cdot, \cdot \rangle : X \times X \to \mathbb{C}$ (or $\mathbb{R}$)

(ii) $\langle\, x, y\,\rangle = \overline{\langle\, y, x\,\rangle}$ for $x,\ y \in X$

(iii) $\langle\, \alpha x + \beta y, z\,\rangle = \alpha\langle\, x, z\,\rangle + \beta\langle\, y, z\,\rangle$ for $x,\ y,\ z \in X$ and $\alpha,\ \beta \in \mathbb{C}$

From (ii), the inner product isn't fully linear in its second coordinate, so we sometimes say that $\langle \cdot, \cdot \rangle$ is *sesquilinear* (or one-and-a-half-linear).

(iv) $\langle\, x, x\,\rangle \geq 0$ for any $x \in X$; $\langle\, x, x\,\rangle = 0$ if and only if $x = 0$.

Note that by linearity we have $\langle\, x, 0\,\rangle = 0 = \langle\, 0, x\,\rangle$ for any $x \in X$. Moreover, from (iii), if $\langle\, x, y\,\rangle = 0$ for all $y \in X$, then $x = 0$.

Examples

1. $\mathbb{C}^n$ with its *usual* inner product $\langle\, x, y\,\rangle = \overline{y}^T x = \sum_{k=1}^n x_k \overline{y}_k$.

2. Given continuous functions $f,\ g : [\,a, b\,] \to \mathbb{C}$, the expression $\langle\, f, g\,\rangle = \int_a^b f(x)\,\overline{g(x)}\,dx$ defines an inner product. Given a strictly positive weight function $w : [\,a, b\,] \to \mathbb{R}$, we might also consider $\langle\, f, g\,\rangle = \int_a^b f(x)\,\overline{g(x)}\,w(x)\,dx$.

In the remainder of this section, we'll assume that X is an inner product space over $\mathbb{C}$. As we'll see directly, an inner product induces a *norm* on X by setting

$$\|x\| = \sqrt{\langle\, x, x\,\rangle}\,. \tag{1}$$

In term of our first two examples we have

$$\|x\| = \left(\sum_{k=1}^n |x_k|^2\right)^{1/2} \quad \text{on } \mathbb{C}^n$$

and

$$\|f\| = \left(\int_a^b |f(x)|^2\,dx\right)^{1/2} \quad \text{or} \quad \left(\int_a^b |f(x)|^2\,w(x)\,dx\right)^{1/2} \quad \text{on } C[\,a, b\,].$$

A key step in verifying that equation (1) defines a norm is given by:

The Cauchy-Schwarz Inequality. *In any inner product space,* $\big|\langle\, x, y\,\rangle\big| \le \|x\|\,\|y\|$, *with equality if and only if* x *and* y *are linearly dependent (that is, parallel).*

In terms of our initial examples,

$$\left|\sum_{k=1}^{n} x_k \overline{y}_k\right| \;\le\; \left(\sum_{k=1}^{n} |x_k|^2\right)^{1/2} \left(\sum_{k=1}^{n} |y_k|^2\right)^{1/2}$$

and

$$\left|\int_a^b f(x)\,\overline{g(x)}\,dx\right| \;\le\; \left(\int_a^b |f(x)|^2\right)^{1/2} \left(\int_a^b |g(x)|^2\right)^{1/2}.$$

Proof. By our earlier remarks, we may assume that x, $y \ne 0$. Now, given $\alpha \in \mathbb{C}$, note that

$$\begin{aligned} 0 \le \|x - \alpha y\|^2 &= \langle\, x - \alpha y, x - \alpha y\,\rangle \\ &= \|x\|^2 - \alpha\overline{\langle\, x, y\,\rangle} - \overline{\alpha}\langle\, x, y\,\rangle + |\alpha|^2\|y\|^2 \\ &= \|x\|^2 - 2\,\mathrm{Re}\,(\overline{\alpha}\,\langle\, x, y\,\rangle) + |\alpha|^2\|y\|^2. \end{aligned}$$

In particular, setting $\alpha = \langle\, x, y\,\rangle / \langle\, y, y\,\rangle = \langle\, x, y\,\rangle / \|y\|^2$, leads to

$$0 \le \|x\|^2 \;-\; 2\frac{|\langle\, x, y\,\rangle|^2}{\|y\|^2} \;+\; \frac{|\langle\, x, y\,\rangle|^2}{\|y\|^4}\,\|y\|^2 = \|x\|^2 \;-\; \frac{|\langle\, x, y\,\rangle|^2}{\|y\|^2}.$$

We leave the case for equality as an exercise (just examine the proof closely). $\square$

It follows from the Cauchy-Schwarz inequality that $\dfrac{|\,\mathrm{Re}\langle\, x, y\,\rangle|}{\|x\|\,\|y\|} \le \dfrac{|\langle\, x, y\,\rangle|}{\|x\|\,\|y\|} \le 1$. Thus we can (and will!) define the *angle* θ between (nonzero) vectors x and y by declaring

$$\cos\theta = \frac{\mathrm{Re}\langle\, x, y\,\rangle}{\|x\|\,\|y\|}.$$

(This uniquely defines θ if we insist that $0 \le \theta \le \pi$.) In particular, we say that x and y are *orthogonal* if $\langle\, x, y\,\rangle = 0$. We will sometimes use the shorthand $x \perp y$ to denote that x and y are orthogonal. From our earlier remarks, every vector is orthogonal to the zero vector (and the zero vector is the only vector with this property).

The observation that

$$\|x + y\|^2 \;=\; \|x\|^2 + 2\,\mathrm{Re}\langle\, x, y\,\rangle + \|y\|^2 \tag{2}$$

(which is the law of cosines in disguise!) leads to two important results.

The Pythagorean Theorem. $\|x+y\|^2 = \|x\|^2 + \|y\|^2$ *if and only if* $\langle\, x, y \,\rangle = 0$.

The Triangle Inequality. $\|x+y\| \le \|x\| + \|y\|$.

Proof. From (2) and the Cauchy-Schwarz inequality,

$$\begin{aligned} \|x+y\|^2 &\le \|x\|^2 + 2|\langle\, x, y \,\rangle| + \|y\|^2 \\ &\le \|x\|^2 + 2\|x\|\,\|y\| + \|y\|^2 \\ &= \big(\,\|x\| + \|y\|\,\big)^2. \quad \square \end{aligned}$$

Corollary. $\|x\| = \sqrt{\langle\, x, x \,\rangle}$ *defines a norm on* X.

At the risk of a bit of repetition, let's take another look at the proof of the Cauchy-Schwarz inequality.

Lemma. *Let* $x,\ y \in X$ *with* $y \ne 0$ *and let* $\alpha = \langle\, x, y \,\rangle / \langle\, y, y \,\rangle$. *Then:*

(i) $(x - \alpha y) \perp y$ *and, of course,* $x = (x - \alpha y) + \alpha y$. *Thus,* x *can be written as the sum of a vector orthogonal to* y *and a vector parallel to* y.

(ii) $\|x\|^2 = \|x - \alpha y\|^2 + |\alpha|^2 \|y\|^2$. *In particular, note that* $\|x\| \ge \|x - \alpha y\|$.

(iii) $\|x - \alpha y\| < \|x - \beta y\|$ *for all* $\beta \ne \alpha$. *Thus,* $y^* = \alpha y$ *is the* unique point in $\operatorname{span} y$ nearest to x*; it is characterized by the requirement that* $(x - y^*) \perp \operatorname{span} y$.

Proof. (i) follows by design; that is, α is chosen to satisfy $\langle\, x - \alpha y, y \,\rangle = 0$. (ii) follows easily from (i) and the Pythagorean theorem. Likewise, (iii) follows from (i) and the Pythagorean theorem. Indeed, from (i) and the linearity of the inner product, it follows that $x - \alpha y$ is perpendicular to every multiple of y; thus, given $\beta \in \mathbb{C}$, we have

$$\begin{aligned} \|x - \beta y\|^2 &= \|(x - \alpha y) + (\beta - \alpha) y\|^2 \\ &= \|x - \alpha y\|^2 + |\beta - \alpha|^2 \|y\|^2 \\ &> \|x - \alpha y\|^2 \end{aligned}$$

unless $\beta = \alpha$. $\square$

The vector $y^* = \alpha y$ in our previous Lemma is called the *orthogonal projection* of x along y; it is the shadow of x onto $\operatorname{span} y$ along a perpendicular "line of sight." In calculus, it is often called the *component* of x in the direction of y.

As we'll see directly, if we inductively apply the Lemma to a *basis* for a (finite-dimensional) subspace Y of X, we'll arrive at a technique for building an *orthogonal basis* (that is, a basis consisting of mutually orthogonal vectors). The benefit in having an orthogonal basis is illustrated by our next result.

Lemma. *Let $\{e_1, \ldots, e_n\}$ be a set of nonzero, mutually orthogonal vectors in X and let $E = \operatorname{span}\{e_1, \ldots, e_n\}$. Then*

(i) *$\{e_1, \ldots, e_n\}$ is* linearly independent*; in fact, $z \in E$ if and only if $z = \sum_{i=1}^n \frac{\langle z, e_i \rangle}{\langle e_i, e_i \rangle} e_i$. Thus, $\{e_1, \ldots, e_n\}$ is a* basis *for E.*

(ii) *Given any $x \in X$, the vector $x - \sum_{i=1}^n \frac{\langle x, e_i \rangle}{\langle e_i, e_i \rangle} e_i$ is* orthogonal *to E.*

(iii) *$e^* = \sum_{j=1}^n \frac{\langle x, e_j \rangle}{\langle e_j, e_j \rangle} e_j$ is the* unique nearest point *to x in E.*

(iv) *$x \in E$ if and only if $\|x\|^2 = \sum_{j=1}^n \frac{|\langle x, e_j \rangle|^2}{\langle e_j, e_j \rangle} = \|e^*\|^2$.*

Proof. To begin, if we set $z = \sum_{i=1}^n \alpha_i e_i$, then

$$\langle z, e_k \rangle = \sum_{i=1}^n \alpha_i \langle e_i, e_k \rangle = \alpha_i \langle e_i, e_i \rangle,$$

which proves (i). Next, given $x \in X$, set $y = x - \sum_{i=1}^n \frac{\langle x, e_i \rangle}{\langle e_i, e_i \rangle} e_i$. A similar calculation then yields

$$\langle y, e_k \rangle = \langle x, e_k \rangle - \langle x, e_k \rangle = 0,$$

and it follows that y is orthogonal to every vector in E. Parts (iii) and (iv) are now almost immediate. Indeed, from (ii), $x - e^*$ is orthogonal to E. Thus, given any vector $e \in E$ we have $e - e^* \in E$ and, hence,

$$\|x - e\|^2 = \|(x - e^*) + (e - e^*)\|^2 = \|x - e^*\|^2 + \|e - e^*\|^2 > \|x - e^*\|^2,$$

unless $e = e^*$. Finally, $x \in E$ if and only if $x = e^*$. But, from (ii), we have $\|x\|^2 = \|x - e^*\|^2 + \|e^*\|^2$. Thus, $x = e^*$ if and only if $\|x\|^2 = \|e^*\|^2 = \sum_{j=1}^n \frac{|\langle x, e_j \rangle|^2}{\langle e_j, e_j \rangle}$. $\square$

The vector e^* of our previous Lemma is the *orthogonal projection* of x onto E. Again, it is characterized by the requirement that $(x - e^*) \perp E$ (as in (ii) of the Lemma).

Again, let's pause to examine our first two examples.

Examples

1. The usual basis $e_k = (0, \ldots, 0, 1, 0, \ldots, 0)$, where the single nonzero entry is in the k-th coordinate, is an *orthonormal basis* for $\mathbb{C}^n$. That is, the e_k are not only mutually orthogonal ($\langle e_i, e_j \rangle = 0$ for $i \neq j$) but also norm one ($\langle e_j, e_j \rangle = 1$ for all j). Note that every vector $x = (x_1, \ldots, x_n) = \sum_{k=1}^n \langle x, e_k \rangle e_k$ has norm $\|x\|^2 = \sum_{k=1}^n |x_k|^2$.

2. Relative to the inner product $\langle f, g \rangle = \frac{1}{2\pi} \int_0^{2\pi} f(x)\, \overline{g(x)}\, dx$, the vectors $\{1, e^{ix}, \ldots, e^{inx}\}$ are orthonormal. Indeed,

$$\int_0^{2\pi} e^{imx}\, \overline{e^{inx}}\, dx = \int_0^{2\pi} e^{i(m-n)x}\, dx = \begin{cases} 0, & \text{if } m \neq n \\ 2\pi, & \text{if } m = n. \end{cases}$$

In the case of real scalars, real-valued functions, and the inner product $\langle f, g \rangle = \frac{1}{\pi} \int_0^{2\pi} f(x)\, g(x)\, dx$, the vectors $\{1/\sqrt{2}, \cos x, \sin x, \ldots, \cos nx, \sin nx\}$ are orthonormal.

We next show how the ideas developed in our first two lemmas can be used to construct orthogonal sequences.

The Gram-Schmidt Process. *Let E be a subspace of X with basis $\{x_1, \ldots, x_n\}$. Then we can find an* orthogonal *basis $\{e_1, \ldots, e_n\}$ for E satisfying*

(i) $\operatorname{span}\{e_1, \ldots, e_k\} = \operatorname{span}\{x_1, \ldots, x_k\}$ *for $k = 1, \ldots, n$, and*

(ii) $0 < \|e_k\| \le \|x_k\|$ *for $k = 1, \ldots, n$.*

Proof. To begin, set $e_1 = x_1 \neq 0$ and let $e_2 = x_2 - \frac{\langle x_2, e_1 \rangle}{\langle e_1, e_1 \rangle} e_1$. Because $x_2 \notin \operatorname{span}\{e_1\} = \operatorname{span}\{x_1\}$, we have $e_2 \neq 0$. Of course, $e_2 \in \operatorname{span}\{e_1, x_2\} = \operatorname{span}\{x_1, x_2\}$ and so we have $\operatorname{span}\{e_1, e_2\} \subset \operatorname{span}\{x_1, x_2\}$. But, from the previous Lemma, e_2 is orthogonal to

$\text{span}\{e_1\} = \text{span}\{x_1\}$. In particular, e_1 and e_2 are linearly independent and it follows that $\text{span}\{e_1, e_2\} = \text{span}\{x_1, x_2\}$. To see that e_2 satisfies (ii), notice that $x_2 = e_2 + \alpha e_1$ and, hence, $\|x_2\|^2 = \|e_2\|^2 + |\alpha|^2 \|e_1\|^2 \geq \|e_2\|^2$.

Continue by induction: Assuming that $\{e_1, \ldots, e_k\}$ have been chosen, set $e_{k+1} = x_{k+1} - \sum_{j=1}^{k} \frac{\langle x_{k+1}, e_j \rangle}{\langle e_j, e_j \rangle} e_j$. Because $x_{k+1} \notin \text{span}\{x_1, \ldots, x_k\} = \text{span}\{e_1, \ldots, e_k\}$, we have $e_{k+1} \neq 0$. Also, $\text{span}\{e_1, \ldots, e_k, e_{k+1}\} \subset \text{span}\{x_1, \ldots, x_k, x_{k+1}\}$. But, as before, e_{k+1} is orthogonal to $\text{span}\{e_1, \ldots, e_k\}$ and, hence, $\{e_1, \ldots, e_k, e_{k+1}\}$ are linearly independent. Thus we must have $\text{span}\{e_1, \ldots, e_k, e_{k+1}\} = \text{span}\{x_1, \ldots, x_k, x_{k+1}\}$. Finally, $\|x_{k+1}\|^2 = \|e_{k+1}\|^2 + \sum_{j=1}^{n} |\alpha_j|^2 \|e_j\|^2 \geq \|e_{k+1}\|^2$. $\square$

Clearly, once we have an orthogonal basis, we simply normalize to arrive at an orthonormal basis (alternatively, by slightly altering the process outlined above, we could construct the vectors e_k to have norm one).

Corollary. *If E is a finite-dimensional subspace of X, then E has an orthonormal basis $\{e_1, \ldots, e_n\}$.*

All of these ideas make very short work of:

The Projection Theorem. *Let E be a subspace of X with* orthonormal *basis $\{e_1, \ldots, e_n\}$. Define $P : X \to X$ by $P(x) = \sum_{k=1}^{n} \langle x, e_k \rangle e_k$ for $x \in X$. Then*

(i) *$P(x) \in E$ and $(x - P(x)) \perp E$; hence, $\|x\|^2 = \|x - P(x)\|^2 + \|P(x)\|^2$.*

(ii) *$P(x)$ is the unique nearest point to x in E.*

(iii) *P is a* projection*; that is, $P^2 = P$.*

(iv) *P is linear, continuous, and satisfies $\|P(x)\| \leq \|x\|$ for all $x \in X$.*

P is called the *orthogonal projection* onto E or, sometimes, the *nearest point map* on E. It follows from (ii) that P is actually independent of the choice of basis.

Hadamard's Inequality

As an application of the ideas in this section, we next present a classical matrix inequality due to Hadamard in 1893.

Theorem. *Let $A = [\, a_{ij} \,]$ be a real or complex $n \times n$ matrix and let $A_j = (a_{ij})_{i=1}^{n}$ denote the j-th* column *of A. Then*

$$|\det A\,| \;\leq\; \prod_{j=1}^{n} \|A_j\| \;=\; \prod_{j=1}^{n} \left(\sum_{i=1}^{n} |a_{ij}|^2 \right)^{1/2}.$$

Equality can only occur if one of the columns $A_j = 0$ or if the columns are orthogonal*; i.e., $\langle\, A_j, A_k \,\rangle = 0$ for $j \neq k$.*

It is well known that $|\det A\,|$ represents the *volume* of the parallelepiped with *edges* (A_j). Thus, Hadamard's inequality states that the volume is *maximal* when the edges are *orthogonal.*

Proof. We may certainly suppose that $\det A \neq 0$, in which case the columns (A_j) are linearly independent. Thus we may apply the Gram-Schmidt process to orthogonalize them, arriving at vectors (b_j) satisfying $b_1 = A_1$,

$$b_j \;=\; A_j - \sum_{i=1}^{j-1} \frac{\langle\, A_j, b_j \,\rangle}{\langle\, b_j, b_j \,\rangle}\, b_i \qquad (j \geq 2), \tag{3}$$

and $\|b_j\| \leq \|A_j\|$ for all j. In particular, the matrix $B = [\, b_1 \cdots b_n \,]$ with columns (b_j) can be obtained from A by *elementary column operations.* It follows that $\det B = \det A$.

But the columns of B are orthogonal, so

$$B^*B \;=\; \operatorname{diag}\big(\, \|b_1\|^2, \ldots, \|b_n\|^2 \,\big)$$

and, hence, $\det(B^*B) = \prod_{j=1}^{n} \|b_j\|^2$. On the other hand, $\det(B^*B) = \overline{\det(B^T)}\, \det B = |\det B\,|^2$. Consequently,

$$|\det A\,| \;=\; |\det B\,| \;=\; \Big(\det(B^*B)\Big)^{1/2} \;=\; \prod_{j=1}^{n} \|b_j\| \;\leq\; \prod_{j=1}^{n} \|A_j\|.$$

Equality can only occur if $\|b_j\| = \|A_j\|$ for all j, which can only occur if $b_j = A_j$ for all j; that is, if and only if the (A_j) were already orthogonal. $\square$

Corollary. *Suppose that A is an $n \times n$ matrix with real or complex entries satisfying $|a_{ij}| \leq 1$ for all i, j. Then $|\det A| \leq n^{n/2}$ with equality if and only if $|a_{ij}| = 1$ for all i, j and the columns of A are orthogonal.*

Proof. In the notation of the previous Theorem, we have $\|A_j\| \leq \sqrt{n}$, thus $|\det A| \leq n^{n/2}$. Equality would mean that the A_j are orthogonal and it would also mean that $\|A_j\| = \sqrt{n}$, which forces $|a_{ij}| = 1$ for all i, j. $\square$

An $n \times n$ real matrix having entries $a_{ij} = \pm 1$ and orthogonal columns is called a *Hadamard matrix*. For example, $A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ is a 2×2 Hadamard matrix. As it happens, it's very easy to construct Hadamard matrices of order 2^n. (And not quite so easy—nor, indeed, always possible—to construct Hadamard matrices of other orders. For example, there is no Hadamard matrix of order 3.) The key is the so-called *Kronecker product*, which we will define by example: The Kronecker product of A with itself is

$$A \otimes A = \begin{bmatrix} 1 \cdot A & 1 \cdot A \\ 1 \cdot A & -1 \cdot A \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \equiv B,$$

which is a 4×4 Hadamard matrix. The product $A \otimes B$ would then yield an 8×8 Hadamard matrix, and so on.

Problem Set 5

Throughout, X denotes a finite-dimensional inner product space over $\mathbb{C}$ with inner product $\langle \cdot, \cdot \rangle$ and with associated norm $\|x\| = \sqrt{\langle x, x \rangle}$.

1. Show that $\big|\langle x, y \rangle\big| = \|x\|\,\|y\|$ if and only if x and y are linearly dependent.

2. Show that $\|x+y\| = \|x\| + \|y\|$ if and only if one of x or y is a *nonnegative* multiple of the other.

3. Show that the *parallelogram law*: $\|x+y\|^2 + \|x-y\|^2 = 2\big(\,\|x\|^2 + \|y\|^2\,\big)$ holds for any x, $y \in X$.

4. Show that the *polarization identity*: $4\langle x, y \rangle = \|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2$ holds for any x, $y \in X$.

5. Show that a linear map $T : X \to Y$ between inner product spaces X and Y is an isometry (into) if and only if it preserves inner products. That is, $\|Tx\| = \|x\|$ for all $x \in X$ if and only if $\langle Tx, Ty \rangle = \langle x, y \rangle$ for all x, $y \in X$. In short, a linear map preserves distances if and only if it preserves angles.

6. Let E be a subspace of X and let $x_0 \in X$. Show that the nearest point to x_0 in E can be characterized as the (unique) point $y_0 \in E$ satisfying $\operatorname{Re}\langle x_0 - y_0, y - y_0 \rangle \le 0$ for all $y \in X$.

7. Consider $C[-1,1]$ with the inner product $\langle f, g \rangle = \int_{-1}^{1} f(x)\,g(x)\,dx$ (where we consider real-valued functions and real scalars).

(i) Apply the Gram-Schmidt process to the linearly independent set $\{1, x, x^2\}$ to find the first three *Legendre polynomials* $P_1(x) = 1$, $P_2(x) = x$, and $P_3(x) = x^2 - \frac{1}{3}$.

(ii) Compute $\min\left\{\int_{-1}^{1} \big|\, e^x - (ax^2 + bx + c)\,\big|^2\, dx : a,\ b,\ c \text{ real}\right\}$.

8. Given a subset A of X, let $A^\perp = \{x \in X : \langle x, a \rangle = 0 \text{ for all } x \in A\}$. The set $A^\perp$ is called the *orthogonal complement* of A. Verify the following for subsets A, B of X.

(a) $A^\perp$ is a subspace of X and $A \cap A^\perp \subset \{0\}$.

(b) $A \subset B^\perp \iff A^\perp \supset B$, and $A \subset B \Longrightarrow A^\perp \supset B^\perp$.

(c) $A \subset A^{\perp\perp}$ and, hence, $\operatorname{span}(A) \subset A^{\perp\perp}$. (In fact, for X finite-dimensional, we actually have $\operatorname{span}(A) = A^{\perp\perp}$.)

9. Let E be a subspace of X and let $P : X \to X$ be the orthogonal projection onto E. Show that $I - P$ is the orthogonal projection onto $E^\perp$ (where I denotes the identity map on X).

10. Let A be an $m \times n$ matrix over $\mathbb{C}$ and let $A^* = \overline{A}^T$ denote the *conjugate transpose* of A. Given $x \in \mathbb{C}^n$ and $y \in \mathbb{C}^m$, show that $\langle Ax, y \rangle = \langle x, A^* y \rangle$. Moreover, this equation characterizes A^*; that is, if B is an $n \times m$ matrix over $\mathbb{C}$ which satisfies $\langle Ax, y \rangle = \langle x, By \rangle$ for all x, $y \in X$, show that $B = A^*$.

11. Let A be an $m \times n$ matrix over $\mathbb{C}$.

(a) Prove that $(\operatorname{range} A)^\perp = \ker(A^*)$.

(b) If $b \in \mathbb{C}^m$ is such that the equation $Ax = b$ has no solution, we can always find a vector $x_0 \in \mathbb{C}^n$ that minimizes $\|b - Ax\|$ over all $x \in \mathbb{C}^n$. Prove that x_0 satisfies $(b - Ax_0) \perp \operatorname{range} A$.

(c) Prove that x_0 satisfies the so-called *normal equation*: $A^*Ax = A^*b$. (Note that A^*A is Hermitian. If A has rank n, then A^*A will be invertible and we can use it to solve for $x_0 = (A^*A)^{-1}A^*b$.)

12. Let $\mathbf{M}_n(\mathbb{C})$ denote the collection of all $n \times n$ matrices with complex entries. $\mathbf{M}_n(\mathbb{C})$ is a vector space over $\mathbb{C}$ under "coordinatewise" addition and scalar multiplication. Show that the expression $\langle A, B \rangle = \operatorname{trace}(B^*A)$ defines an inner product on $\mathbf{M}_n(\mathbb{C})$.

13. Let $A \in \mathbf{M}_n(\mathbb{C})$.

(a) Show that $[\, x, y \,] = \langle Ax, y \rangle$ defines an inner product on $\mathbb{C}^n$ if and only if A satisfies:

(i) $A^* = A$. (We say that A is *Hermitian* or self-adjoint if $A^* = A$.)

(ii) $\langle Ax, x \rangle \geq 0$ for any x, and $\langle Ax, x \rangle = 0$ only for $x = 0$. (In other words, the quadratic form $f(x, y) = \langle Ax, y \rangle$ is positive definite; this implies that A has strictly positive [real] eigenvalues.)

(b) Conversely, show that any sesquilinear, positive definite quadratic form on $\mathbb{C}^n$ is of the form $\langle Ax, y \rangle$ for some Hermitian matrix A.

Hilbert's Double Series Theorem

Theorem. *If (a_m) and (b_n) are real, square-summable sequences, then*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{m+n} \;<\; \pi\left(\sum_{m=1}^{\infty} a_m^2\right)^{1/2}\left(\sum_{n=1}^{\infty} b_n^2\right)^{1/2} \tag{1}$$

unless one of (a_n) or (b_m) is identically zero. The constant π is best possible.

It's unusual to see a strict inequality with a sharp bound! Hilbert proved this inequality, with constant 2π, in his famous series of lectures on integral equations (roughly, 1904–1911); his result was first published by Weyl in 1908. The best constant, together with various generalizations, was provided by Schur in 1911. We will present two proofs, along with a generalization to integral inequalities.

Our first approach uses the Cauchy-Schwarz inequality in the form:

$$\sum_{k\in I} x_k y_k \;\le\; \left(\sum_{k\in I} x_k^2\right)^{1/2}\left(\sum_{k\in I} y_k^2\right)^{1/2}, \tag{2}$$

where (x_k) and (y_k) are real sequences and where I is a countable set; in this case,

$$I = \{(m,n) : m,n = 1,2,3,\ldots\}.$$

This is a relatively straightforward generalization of the $\mathbb{R}^n$ version because

$$\sum_{k\in I} x_k^2 \;=\; \sup\left\{\sum_{k\in J} x_k^2 : J\subset I,\; J \text{ finite}\right\}.$$

That is, (2) follows from the finite-sum version of Cauchy-Schwarz by showing that all (finite length) partial sums satisfy (2).

The idea behind our first proof of Hilbert's theorem is to take advantage of the symmetry of the summand, relative to m and n. We begin by rewriting:

$$\frac{a_m b_n}{m+n} \;=\; \frac{a_m}{\sqrt{m+n}}\left(\frac{m}{n}\right)^{\lambda}\cdot\frac{b_n}{\sqrt{m+n}}\left(\frac{n}{m}\right)^{\lambda},$$

where, if possible, λ will be chosen so that each of the factors on the right is square-summable (over the index set I). Applying Cauchy-Schwarz we get:

$$\left(\sum_{m,n} \frac{a_m b_n}{m+n}\right)^2 \leq \sum_{m,n} \frac{a_m^2}{m+n} \left(\frac{m}{n}\right)^{2\lambda} \sum_{m,n} \frac{b_n^2}{m+n} \left(\frac{n}{m}\right)^{2\lambda}. \tag{3}$$

But

$$\sum_{m,n} \frac{a_m^2}{m+n} \left(\frac{m}{n}\right)^{2\lambda} = \sum_{m=1}^{\infty} a_m^2 \sum_{n=1}^{\infty} \frac{1}{m+n} \left(\frac{m}{n}\right)^{2\lambda}$$

and, similarly,

$$\sum_{m,n} \frac{b_n^2}{m+n} \left(\frac{n}{m}\right)^{2\lambda} = \sum_{n=1}^{\infty} b_n^2 \sum_{m=1}^{\infty} \frac{1}{m+n} \left(\frac{n}{m}\right)^{2\lambda}.$$

Thus, we'll have a proof of Hilbert's inequality (with some constant) if we can find a finite bound B_λ such that

$$\sum_{n=1}^{\infty} \frac{1}{m+n} \left(\frac{m}{n}\right)^{2\lambda} \leq B_\lambda$$

for all m; that is, B_λ may depend on λ, but not on m.

Now for $\lambda > 0$ (and any m), the sequence $\frac{1}{m+n}\left(\frac{m}{n}\right)^{2\lambda}$ *decreases* as n increases, so we can appeal to the integral test:

Lemma. *If $f : [0,\infty) \to [0,\infty)$ is strictly decreasing, then*

$$\int_1^{\infty} f(x)\,dx < \sum_{n=1}^{\infty} f(n) < \int_0^{\infty} f(x)\,dx.$$

Returning to our search for the bound B_λ, we find

$$\sum_{n=1}^{\infty} \frac{1}{m+n} \left(\frac{m}{n}\right)^{2\lambda} < \int_0^{\infty} \frac{1}{m+x} \left(\frac{m}{x}\right)^{2\lambda} dx = \int_0^{\infty} \frac{1}{1+y} \frac{1}{y^{2\lambda}}\, dy.$$

As we've seen (in a slightly different form), this integral exists provided that $0 < 2\lambda < 1$ and, in fact, equals $\Gamma(2\lambda)\Gamma(1-2\lambda)$; that is, we can take $B_\lambda = \Gamma(2\lambda)\Gamma(1-2\lambda)$. In other words, we've just shown that

$$\sum_{m,n} \frac{a_m b_n}{m+n} < B_\lambda \left(\sum_{m=1}^{\infty} a_m^2\right)^{1/2} \left(\sum_{n=1}^{\infty} b_n^2\right)^{1/2}.$$

But we've also seen that the minimum value of B_λ occurs when $\lambda = 1/4$, in which case $B_{1/4} = (\Gamma(1/2)^2 = \pi$. This proves Hilbert's theorem, save the claim that π is best possible.

Corollary. *Let* (a_m) *and* (b_n) *be nonnegative, let* $1 < p < \infty$*, and let* q *satisfy* $\frac{1}{p} + \frac{1}{q} = 1$*. Then*

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_m b_n}{m+n} \;<\; \Gamma(1/p)\Gamma(1/q) \left(\sum_{m=1}^{\infty} a_m^p \right)^{1/p} \left(\sum_{n=1}^{\infty} b_n^q \right)^{1/q}$$

unless one of (a_n) *or* (b_m) *is identically zero.*

Proof. In this case, we write

$$\frac{1}{m+n} \;=\; \left[\frac{1}{m+n} \left(\frac{m}{n} \right)^{1/q} \right]^{1/p} \left[\frac{1}{m+n} \left(\frac{n}{m} \right)^{1/p} \right]^{1/q}$$

and apply Hölder's inequality, which will lead to the bounds

$$\int_0^{\infty} \frac{1}{1+y} \frac{1}{y^{1/p}}\, dy = \Gamma(1/p)\Gamma(1/q) \quad \text{and} \quad \int_0^{\infty} \frac{1}{1+y} \frac{1}{y^{1/q}}\, dy = \Gamma(1/q)\Gamma(1/p). \qquad \square$$

As it happens, $\Gamma(x)\Gamma(1-x) = \pi/(\sin \pi x)$ for $0 < x < 1$ and so the bound in the Corollary can be written as $\pi/(\sin(\pi/p))$, which is actually best possible.

The integral version of Hilbert's inequality can be proved in essentially the same way as the discrete version and, in fact, will yield the discrete version as a corollary.

Theorem. *Let* $f,\, g : [0, \infty) \to [0, \infty)$*, let* $1 < p < \infty$*, and let* $\frac{1}{p} + \frac{1}{q} = 1$*. Then*

$$\int_0^{\infty} \int_0^{\infty} \frac{f(x)\, g(y)}{x+y}\, dx\, dy \;<\; \frac{\pi}{\sin(\pi/p)} \left(\int_0^{\infty} f(x)^p\, dx \right)^{1/p} \left(\int_0^{\infty} g(y)^p\, dy \right)^{1/p}$$

unless one of f *or* g *is identically zero. The constant* $\pi/\sin(\pi/p)$ *is best possible.*

In this case we would write

$$\frac{1}{x+y} = \left[\frac{1}{x+y} \left(\frac{x}{y} \right)^{1/q} \right]^{1/p} \left[\frac{1}{x+y} \left(\frac{y}{x} \right)^{1/p} \right]^{1/q}$$

and proceed as before. Instead of completing this calculation, let's opt for a slightly more general theorem:

Theorem. *Let* $K : [0,\infty)\times[0,\infty)\to[0,\infty)$ *satisfy* $K(\lambda x,\lambda y)=\lambda^{-1}K(x,y)$ *for all* $\lambda>0$. *Then, for* f, g, *and* p *as above,*

$$\int_0^\infty\int_0^\infty K(x,y)\,f(x)\,g(y)\,dx\,dy \;\le\; C\left(\int_0^\infty f(x)^p\,dx\right)^{1/p}\left(\int_0^\infty g(y)^p\,dy\right)^{1/p}. \qquad (4)$$

The constant C *is given by the common value*

$$C=\int_0^\infty K(x,1)\,x^{-1/p}\,dx=\int_0^\infty K(1,y)\,y^{-1/q}\,dy. \qquad (5)$$

If $K>0$, *then the inequality is strict unless one of* f *or* g *is identically zero.*

Proof. We begin with the change of variable $y=ux$ (and a few changes in the order of integration).

$$\begin{aligned} I=\int_0^\infty\int_0^\infty K(x,y)\,f(x)\,g(y)\,dx\,dy &= \int_0^\infty f(x)\left[\int_0^\infty K(x,y)\,g(y)\,dy\right]dx \\ &= \int_0^\infty f(x)\left[\int_0^\infty xK(x,ux)\,g(ux)\,du\right]dx \\ &= \int_0^\infty f(x)\left[\int_0^\infty K(1,u)\,g(ux)\,du\right]dx \\ &= \int_0^\infty K(1,u)\left[\int_0^\infty f(x)\,g(ux)\,dx\right]du. \end{aligned}$$

We now apply Hölder's inequality to the inner integral, using the same change of variable $y=ux$:

$$\int_0^\infty f(x)\,g(ux)\,dx \le \left(\int_0^\infty |f(x)|^p\,dx\right)^{1/p}\cdot u^{-1/q}\left(\int_0^\infty |g(y)|^q\,dy\right)^{1/q}.$$

Thus,

$$I\le\int_0^\infty K(1,u)\,u^{-1/q}\,du\left(\int_0^\infty |f(x)|^p\,dx\right)^{1/p}\left(\int_0^\infty |g(y)|^q\,dy\right)^{1/q}.$$

The case for strict inequality in (4) follows from a careful examination of the case for equality in Hölder's inequality (which we will forego). The fact that the two integrals in (5) are equal is left as an exercise. $\square$

Now the integral version of Hilbert's inequality can be used to deduce the discrete version; indeed, because we have

$$\int_{m-1}^m\int_{n-1}^n \frac{dy\,dx}{x+y} \;\ge\; \frac{1}{m+n},$$

we could define $f(x) = a_m$ for $m - 1 \le x < m$, $g(y) = b_n$, $n - 1 \le y < n$, and write the double sum in (1) as a double integral over $[0, \infty)^2$. But we can actually do a bit better, arriving at an even sharper version of (1). To see this, note that the function $h(\alpha) = (m + n - 1 + \alpha)^{-1}$ is strictly convex for $-1 \le \alpha \le 1$; thus,

$$\frac{1}{m+n-1-\alpha} + \frac{1}{m+n-1+\alpha} \; > \; \frac{2}{m+n-1}.$$

Now watch closely!

$$\int_{m-1}^{m}\int_{n-1}^{n} \frac{dy\,dx}{x+y} \; = \; \int_{-1/2}^{1/2}\int_{-1/2}^{1/2} \frac{dy\,dx}{m+n-1+x+y} \; = \; \int_{-1/2}^{1/2}\int_{-1/2}^{1/2} \frac{dy\,dx}{m+n-1-x-y}.$$

(Where we first exchanged x and y for $x + m - 1/2$ and $y + n - 1/2$, then exchanged x and y for $-x$ and $-y$.) All three integrals are equal and the second two average to be strictly bigger than $(m + n - 1)^{-1}$; thus we have

$$\int_{m-1}^{m}\int_{n-1}^{n} \frac{dy\,dx}{x+y} \; > \; \frac{1}{m+n-1}.$$

Replacing m and n by $m + 1$ and $n + 1$ leads to the following sharper version of Hilbert's inequality.

Corollary. *If (a_m) and (b_n) are real, square-summable sequences, then*

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{a_m b_n}{m+n+1} \; < \; \pi \left(\sum_{m=0}^{\infty} a_m^2 \right)^{1/2} \left(\sum_{n=0}^{\infty} b_n^2 \right)^{1/2}$$

unless one of (a_n) or (b_m) is identically zero.

We now apply our Corollary to the *moment sequence* of a square-integrable function. For this we'll find it helpful to have a converse to Hölder's inequality, a result that's useful in its own right.

Lemma. *Let $1 < p < \infty$ and let $\frac{1}{p} + \frac{1}{q} = 1$. Suppose we're given a real or complex sequence (a_n) and a positive constant C that satisfy:*

$$\sum_{n=1}^{\infty} a_n b_n \le C \left(\sum_{n=1}^{\infty} |b_n|^q \right)^{1/q}$$

whenever (b_n) is a q-th power summable sequence. Then (a_n) is p-th power summable and, moreover, $\sum_{n=1}^{\infty} |a_n|^p \le C^p$.

Proof. It suffices to show that $\sum_{n=1}^{N} |a_n|^p \le C^p$ for all N. But setting $b_n = |a_n|^{p-1}\mathrm{sgn}\, a_n$, for $n = 1, \ldots, N$, and $b_n = 0$ otherwise, we have $|b_n|^q = |a_n|^p$, for $n = 1, \ldots, N$. Thus,

$$\sum_{n=1}^{N} a_n b_n = \sum_{n=1}^{N} |a_n|^p \le C \left(\sum_{n=1}^{N} |a_n|^p \right)^{1/q}.$$

Dividing by $\left(\sum_{n=1}^{N} |a_n|^p\right)^{1/q}$ (which we may suppose is nonzero), we get $\left(\sum_{n=1}^{N} |a_n|^p\right)^{1/p} \le C$. $\square$

Corollary. *Let $f : [0,1] \to \mathbb{R}$ be square-integrable and nonzero. For each $n = 0, 1, 2, \ldots$, define $a_n = \int_0^1 x^n f(x)\, dx$. Then*

$$\sum_{n=0}^{\infty} a_n^2 \;\le\; \pi \int_0^1 f(x)^2\, dx.$$

The constant π is best possible.

Proof. We may suppose that $f \ge 0$. Indeed,

$$|a_n| \le \int_0^1 x^n |f(x)|\, dx = b_n$$

(the moment sequence for $|f|$), so it would suffice to consider $|f|$.

Now if (b_n) is any square-summable sequence, then, for all N, we have

$$\begin{aligned}
\sum_{n=0}^{N} a_n b_n &= \sum_{n=0}^{N} b_n \int_0^1 x^n f(x)\, dx = \int_0^1 f(x) \sum_{n=0}^{N} b_n x^n\, dx \\
&\le \left(\int_0^1 f(x)^2\, dx \right)^{1/2} \left(\int_0^1 \left(\sum_{n=0}^{N} b_n x^n \right)^2 dx \right)^{1/2} \\
&= \left(\int_0^1 f(x)^2\, dx \right)^{1/2} \left(\sum_{m=0}^{N} \sum_{n=0}^{N} \frac{b_m b_n}{m+n+1} \right)^{1/2} \\
&< \left(\pi \int_0^1 f(x)^2\, dx \right)^{1/2} \left(\sum_{n=0}^{N} b_n^2 \right)^{1/2}.
\end{aligned}$$

From the converse to Hölder's inequality (or, in this case, the Cauchy-Schwarz inequality), it follows that (a_n) is square-summable and satisfies $\sum_n a_n^2 \leq \pi \int_0^1 f(x)^2\,dx$. The fact that π is the best constant follows from considering the function $f(x) = (1-x)^{-\varepsilon-\frac{1}{2}}$ (and letting $\varepsilon \to 0$), a calculation we will omit. $\square$

Our second proof of Hilbert's inequality has the advantage of being entirely elementary, but has the disadvantage of giving a slightly weaker result (given our present state of knowledge). The proof, due to Toeplitz in 1910, begins with a simple observation.

Lemma. $\dfrac{i}{2\pi}\displaystyle\int_0^{2\pi}(t-\pi)\,e^{int}\,dt = \dfrac{1}{n}$ *for* $n \in \mathbb{Z}$, $n \neq 0$.

Proof. Integrate by parts:

$$\frac{i}{2\pi}\int_0^{2\pi}(t-\pi)\,e^{int}\,dt = \frac{i}{2\pi}\int_0^{2\pi}(t-\pi)\,d\left(\frac{1}{in}\,e^{int}\right) = \frac{1}{2\pi n}(t-\pi)\,e^{int}\,\Big]_0^{2\pi} = \frac{1}{n}. \quad \square$$

This simple observation gives us an integral representation for the left-hand side of Hilbert's inequality.

Lemma. $\displaystyle\sum_{m=1}^{N}\sum_{n=1}^{N}\frac{a_m b_n}{m+n} = \frac{i}{2\pi}\int_0^{2\pi}(t-\pi)\,f(t)\,g(t)\,dt$, *where* $f(t) = \displaystyle\sum_{m=1}^{N} a_m e^{imt}$ *and* $g(t) = \displaystyle\sum_{n=1}^{N} b_n e^{int}$.

Finally, we apply the Cauchy-Schwarz inequality (and the fact that the functions e^{int} are orthonormal on $[\,0, 2\pi\,]$) to arrive at:

$$\begin{aligned}
\left|\sum_{m=1}^{N}\sum_{n=1}^{N}\frac{a_m b_n}{m+n}\right| &= \left|\frac{i}{2\pi}\int_0^{2\pi}(t-\pi)\,f(t)\,g(t)\,dt\right| \\
&\leq \pi\cdot\frac{1}{2\pi}\int_0^{2\pi}|f(t)|\,|g(t)|\,dt \\
&\leq \pi\left(\frac{1}{2\pi}\int_0^{2\pi}|f(t)|^2\,dt\right)^{1/2}\left(\frac{1}{2\pi}\int_0^{2\pi}|g(t)|^2\,dt\right)^{1/2} \\
&= \pi\left(\sum_{m=1}^{N} a_m^2\right)^{1/2}\left(\sum_{n=1}^{N} b_n^2\right)^{1/2}.
\end{aligned}$$

Because of the special nature of the representation in our previous Lemma, it's not so clear that this proof will yield the infinite sum version (unless we assume that (a_m) and (b_n) are nonnegative). Moreover, we would be hard pressed to prove strict inequality by this method alone. Nevertheless, Toeplitz's proof is not only simple, but has the benefit of amply demonstrating the interplay between "discrete" and "continuous" inner product spaces.

Our final result in this section is Schur's generalization of Hilbert's inequality from 1911. Surprisingly, the proof we will give is short, elementary, and does not appeal in any way to Hilbert's inequality.

Lemma. *Let c_k be a complex number, let λ_k be an integer, and let $0 < \alpha < 1$. Then*

$$\int_0^{2\pi} \left| \sum_{k=1}^{n} c_k e^{i\lambda_k x} \right| dx \;\geq\; 2\sin\pi\alpha \left| \sum_{k=1}^{n} \frac{c_k}{\lambda_k - \alpha} \right|.$$

Proof. We begin with the estimate

$$\begin{aligned} \int_0^{2\pi} \left| \sum_{k=1}^{n} c_k e^{i\lambda_k x} \right| dx \;&\geq\; \left| \int_0^{2\pi} \sum_{k=1}^{n} c_k e^{i(\lambda_k-\alpha)x}\, dx \right| \\ &= \left| \sum_{k=1}^{n} c_k \int_0^{2\pi} e^{i(\lambda_k-\alpha)x}\, dx \right| \\ &= \left| \sum_{k=1}^{n} \frac{c_k}{\lambda_k-\alpha} \int_0^{2\pi(\lambda_k-\alpha)} e^{iu}\, du \right|. \end{aligned}$$

But the modulus of the last integral depends only on α:

$$\begin{aligned} \int_0^{2\pi(\lambda_k-\alpha)} e^{iu}\, du \;&=\; -i\, e^{iu} \Big]_0^{2\pi(\lambda_k-\alpha)} \\ &= -i\Big[e^{i2\pi(\lambda_k-\alpha)} - 1 \Big] \\ &= -i\Big[e^{-i2\pi\alpha} - 1 \Big] \\ &= -i\, e^{-i\pi\alpha}\Big[e^{-i\pi\alpha} - e^{i\pi\alpha} \Big] \\ &= -2e^{-i\pi\alpha} \sin\pi\alpha. \end{aligned}$$

Thus, $\left| \int_0^{2\pi(\lambda_k-\alpha)} e^{iu}\, du \right| = 2\sin\pi\alpha$ and the result follows. $\square$

Theorem. *Let* a_m *and* b_n *be complex numbers and let* $0 < \alpha < 1$. *Then*

$$\left| \sum_{m=1}^{N} \sum_{n=1}^{N} \frac{a_m b_n}{m+n-\alpha} \right| \;\le\; \frac{\pi}{\sin \pi\alpha} \left(\sum_{m=1}^{N} a_m^2 \right)^{1/2} \left(\sum_{n=1}^{N} b_n^2 \right)^{1/2} .$$

Proof. This is a simple matter of applying the Cauchy-Schwarz inequality and appealing to our previous Lemma. First, we essentially repeat a calculation from Toeplitz's proof:

$$\frac{1}{2\pi} \int_0^{2\pi} \left| \sum_{m=1}^{N} a_m e^{imx} \sum_{n=1}^{N} b_n e^{inx} \right| dx \;\le\; \left(\sum_{m=1}^{N} a_m^2 \right)^{1/2} \left(\sum_{n=1}^{N} b_n^2 \right)^{1/2} . \tag{6}$$

On the other hand, from our previous Lemma,

$$\begin{aligned} \frac{1}{2\pi} \int_0^{2\pi} \left| \sum_{m=1}^{N} a_m e^{imx} \sum_{n=1}^{N} b_n e^{inx} \right| dx \;&=\; \frac{1}{2\pi} \int_0^{2\pi} \left| \sum_{m=1}^{N} \sum_{n=1}^{N} a_m b_n e^{i(m+n)x} \right| dx \\ &\ge\; \frac{1}{2\pi} \cdot 2 \sin \pi\alpha \left| \sum_{m=1}^{N} \sum_{n=1}^{N} \frac{a_m b_n}{m+n-\alpha} \right| \end{aligned} \tag{7}$$

(by taking $c_k = a_m b_n$ and $\lambda_k = m+n$ in our previous Lemma). Combining (6) and (7) completes the proof. $\square$

Hardy's Inequality

In his search for a new proof of Hilbert's double series theorem, Hardy discovered the following inequalities (in 1920, but without the best constant in Theorem 1; this was later rectified by Landau in 1926).

Theorem 1. *Let $1 < p < \infty$, let (a_n) be nonnegative and p-th power summable, and let $A_n = a_1 + a_2 + \cdots + a_n$. Then*

$$\sum_{n=1}^{\infty} \left(\frac{A_n}{n}\right)^p \;<\; \left(\frac{p}{p-1}\right)^p \sum_{n=1}^{\infty} a_n^p \tag{1}$$

unless (a_n) is identically zero. The constant is best possible.

The integral analogue of Theorem 1 is given as

Theorem 2. *Let $1 < p < \infty$, let $f : [0, \infty) \to [0, \infty)$ be p-th power integrable, and let $F(x) = \int_0^x f(t)\, dt$. Then*

$$\int_0^{\infty} \left(\frac{F(x)}{x}\right)^p dx \;<\; \left(\frac{p}{p-1}\right)^p \int_0^{\infty} f(x)^p \, dx \tag{2}$$

unless f is identically zero. The constant is best possible.

Our proof of Theorem 1 is due to Elliot (1926). Our proof of Theorem 2 is (essentially) Hardy's original proof.

It may help to outline the heuristics that led Hardy to Theorem 1. Hardy's approach to the double series theorem was to treat the sums above and below the diagonal as essentially identical; that is, he reasoned that it would suffice to examine

$$\sum_{n=1}^{\infty} \sum_{m=1}^{n} \frac{a_m b_n}{m+n} \;\le\; \sum_{n=1}^{\infty} \sum_{m=1}^{n} \frac{a_m b_n}{n} \;=\; \sum_{n=1}^{\infty} \frac{A_n}{n}\, b_n. \tag{3}$$

Now the conclusion of Hilbert's theorem is, essentially, that the sum on the left converges whenever $\sum_m a_m^p$ and $\sum_n b_n^q$ converge (this by way of an application of Hölder's inequality). But, Hardy reasoned, perhaps the convergence of $\sum_m a_m^p$ already implies the convergence

of $\sum_n (A_n/n)^p$, in which case an application of Hölder's inequality to the right-hand side of (3) would lead to a more direct proof of Hilbert's theorem; whence Theorem 1.

Proof of Theorem 1. The proof begins with an observation that will help us establish strict inequality: We may suppose that $a_1 > 0$. Indeed, if the theorem has been proved in that case, then it will also hold for a sequence (b_n) with $b_1 = 0$ for, in this case, setting $a_n = b_{n+1}$, we would have

$$\begin{aligned}\left(\frac{b_2}{2}\right)^p + \left(\frac{b_2+b_3}{3}\right)^p + \cdots &= \left(\frac{a_1}{2}\right)^p + \left(\frac{a_1+a_2}{3}\right)^p + \cdots \\ &\le \left(\frac{a_1}{1}\right)^p + \left(\frac{a_1+a_2}{2}\right)^p + \cdots \\ &< \left(\frac{p}{p-1}\right)^p \sum_n a_n^p = \left(\frac{p}{p-1}\right)^p \sum_n b_n^p.\end{aligned}$$

We now set $x_n = A_n/n$ and estimate:

$$\begin{aligned}x_n^p - \frac{p}{p-1}\, x_n^{p-1} a_n \;&=\; x_n^p - \frac{p}{p-1}\left\{n x_n - (n-1)x_{n-1}\right\} x_n^{p-1} \\ &=\; \left(1 - \frac{np}{p-1}\right) x_n^p + \frac{(n-1)p}{p-1}\, x_n^{p-1} x_{n-1} \\ &\le\; \left(1 - \frac{np}{p-1}\right) x_n^p + \frac{(n-1)p}{p-1}\left\{\left(\frac{p-1}{p}\right) x_n^p + \frac{1}{p}\, x_{n-1}^p\right\} \qquad (4) \\ &=\; \frac{1}{p-1}\left\{(n-1)x_{n-1}^p - n x_n^p\right\},\end{aligned}$$

where we've used Young's inequality in (4). Next we sum over $n = 1, \ldots, N$, noting that the terms on the right, above, will telescope, and conclude that

$$\sum_{n=1}^N x_n^p - \frac{p}{p-1} \sum_{n=1}^N x_n^{p-1} a_n \;\le\; -\frac{N x_n^p}{p-1} \;\le\; 0.$$

That is,

$$\sum_{n=1}^N x_n^p \;\le\; \frac{p}{p-1} \sum_{n=1}^N x_n^{p-1} a_n.$$

Applying Hölder's inequality then yields

$$\sum_{n=1}^N x_n^p \;\le\; \frac{p}{p-1}\left(\sum_{n=1}^N x_n^p\right)^{1-1/p} \left(\sum_{n=1}^N a_n^p\right)^{1/p}. \qquad (5)$$

Collecting terms and raising both sides to the p-th power gives us the finite version of our result (but with a weak inequality):

$$\sum_{n=1}^{N} x_n^p \;\le\; \left(\frac{p}{p-1}\right)^p \sum_{n=1}^{N} a_n^p.$$

It follows that

$$\sum_{n=1}^{\infty} x_n^p \;\le\; \left(\frac{p}{p-1}\right)^p \sum_{n=1}^{\infty} a_n^p, \tag{6}$$

where both sides of the inequality are finite. Finally, to see that we actually have strict inequality, note that equality in (5) would force (x_n^p) and (a_n^p) to be proportional. But we took $a_1 = x_1 > 0$, so the constant of proportionality would have to be 1. That is, we would have $a_n = A_n/n$ for all n, which can only occur if (a_n) is constant. (And this is consistent with equality in (4), which would force (x_n) to be constant.) This is obviously inconsistent with the fact that (a_n) is p-th power summable; thus, (6) (and (1)) actually holds with strict inequality.

To see that $\{p/(p-1)\}^p$ is the best constant, one approach is to consider the sequence $a_n = n^{-\frac{1}{p}-\varepsilon}$ and let $\varepsilon \to 0$. It's a straightforward estimate, but we'll skip the details. $\square$

We next give (what is essentially) Hardy's proof of Theorem 2. It is very similar in spirit (if not in detail) to the proof we've just given for Theorem 1.

Proof of Theorem 2. As in the proof of Theorem 1, we will first establish a slightly weaker version of (2). In particular, we will first show that

$$\int_0^T \left(\frac{F(x)}{x}\right)^p dx \;\le\; \left(\frac{p}{p-1}\right)^p \int_0^T f(x)^p\, dx \tag{2$'$}$$

for all T sufficiently large. To this end, note that if f is not identically zero, then $\int_0^\infty f^p > 0$ and, hence, $\int_0^T f^p > 0$ for all T sufficiently large. (This expression is analogous to the sum $\sum_{n=1}^N a_n^p$, and we will need to divide by it later in the proof, much as we did in the proof of Theorem 1.)

Next, note that if f is p-th power integrable, then f is integrable over $[0, x]$ for any $x > 0$ and, moreover,

$$F(x)^p = \left(\int_0^x f(t)\,dt\right)^p \leq x^{p-1} \int_0^x f(t)^p\,dt$$

from Hölder's inequality. In particular, we have $x^{1-p}F(x)^p \to 0$ as $x \to 0^+$, a fact that will come in handy momentarily.

We're ready to attack (2'). We first integrate by parts, then apply Hölder's inequality:

$$\begin{aligned}
\int_0^T \left(\frac{F(x)}{x}\right)^p dx &= \frac{1}{1-p}\int_0^T F(x)^p\,d\big(x^{1-p}\big) \\
&= \frac{1}{1-p}\,x^{1-p}F(x)^p\Big]_0^T + \frac{p}{p-1}\int_0^T x^{1-p}F(x)^{p-1} f(x)\,dx \\
&= \frac{p}{p-1}\int_0^T \left(\frac{F(x)}{x}\right)^{p-1} f(x)\,dx - \frac{1}{p-1}\,T^{1-p}F(T)^p \qquad (7) \\
&\leq \frac{p}{p-1}\int_0^T \left(\frac{F(x)}{x}\right)^{p-1} f(x)\,dx \qquad (8) \\
&\leq \frac{p}{p-1}\left(\int_0^T f(x)^p\,dx\right)^{1/p}\left(\int_0^T \left(\frac{F(x)}{x}\right)^p dx\right)^{1-1/p}, \qquad (9)
\end{aligned}$$

where (in (7)) we've used the fact that $x^{1-p}F(x)^p \to 0$ as $x \to 0^+$ and (in (8)) the fact that $F \geq 0$. (Note that if $(F(x)/x)^p$ is integrable over $[0, \infty)$, as the theorem suggests, then we would expect to have $x(F(x)/x)^p \to 0$ as $x \to \infty$. Thus, dropping the term $T^{1-p}F(T)^p$ is unlikely to cause any problems.)

We've been here before! $\int_0^T (F/x)^p$ occurs on both sides of our inequality, but to different powers. Divide, then raise both sides to the p-th power to arrive at (2'):

$$\int_0^T \left(\frac{F(x)}{x}\right)^p dx \;\leq\; \left(\frac{p}{p-1}\right)^p \int_0^T f(x)^p\,dx.$$

Because this inequality holds for all T sufficiently large, we've proved that

$$\int_0^\infty \left(\frac{F(x)}{x}\right)^p dx \;\leq\; \left(\frac{p}{p-1}\right)^p \int_0^\infty f(x)^p\,dx, \qquad (10)$$

which is very nearly the conclusion we were hoping for. All that's missing is strict inequality, which follows from a closer examination of our application of Hölder's inequality

in (9). Indeed, equality in (10) would force equality in (9), which in turn would force $(F(x)/x)^p$ and $f(x)^p$ to be proportional. But this would force f to be a power of x, which is inconsistent with the assumption that f is p-th power integrable over $[0,\infty)$. (If we assume for the moment that f is continuous, then the equation $xf(x) = C\int_0^x f$ would imply that f is differentiable and satisfies the differential equation $xf'(x) = (C-1)f(x)$, which has solution $f(x) = Bx^{C-1}$.)

As with Theorem 1, the proof that the constant $\{p/(p-1)\}^p$ is best would follow from considering, for example, the function $f(x) = x^{-\frac{1}{p}-\varepsilon}$, $x \geq 1$, and $f(x) = 0$, otherwise. Again, we will forego the details. $\square$

The Inequalities of Carleman, Knopp, Jensen, and Carleson

We begin with a simple application of Hardy's Theorem 1.

Corollary. *Let $1 < p < \infty$ and let (a_n) be nonnegative. Then*

$$\sum_{n=1}^{\infty}\left(\frac{a_1^{1/p} + a_2^{1/p} + \cdots + a_n^{1/p}}{n}\right)^p \;\leq\; \left(\frac{p}{p-1}\right)^p \sum_{n=1}^{\infty} a_n.$$

Note that the summand on the left can be written as $M_{1/p}^{(n)}(a)$, where the superscript (n) is a reminder that we're summing only n terms. Recall, too, that $M_0^{(n)}(a) = (a_1a_2\cdots a_n)^{1/n} \leq M_{1/p}^{(n)}(a)$; in fact, if we let $p \to \infty$ (with n still fixed), $M_{1/p}^{(n)}(a)$ decreases to $M_0^{(n)}(a)$ while the constant on the right tends to e. Thus, we have another "name" inequality, due to Carleman in 1923 (but without a proof of strict inequality):

Carleman's Inequality. *For (a_n) nonnegative,*

$$\sum_{n=1}^{\infty}(a_1a_2\cdots a_n)^{1/n} < e\sum_{n=1}^{\infty} a_n \tag{1}$$

provided that (a_n) is not identically zero. The constant is best possible.

An elementary yet elegant proof of Carleman's inequality, due to Pólya in 1926, is outlined in the exercises. We'll give a second proof shortly. The proof that the constant e is best possible will be left for another day.

There is an integral analogue of Carleman's inequality. To find it, we first rewrite (1) as $\sum_{n=1}^{\infty} \exp\left(\frac{1}{n}\sum_{k=1}^{\infty}\log a_k\right) \leq e\sum_{n=1}^{\infty} a_n$, which suggests:

Knopp's Inequality. *If $f : [0,\infty) \to (0,\infty)$ is integrable, then*

$$\int_0^{\infty} \exp\left\{\frac{1}{x}\int_0^x \log\big(f(t)\big)\,dt\right\}dx \;<\; e\int_0^{\infty} f(x)\,dx$$

unless f is identically zero (*and we interpret* $\exp(\log(0)) = 0$).

Curiously, Knopp's original inequality (from 1928) was stated with 1 as the lower limit of integration (in all instances), and that inequality is actually *false!* It fails for the

function $f(x) = 1/x^2$, which can be verified by direct calculation. Nevertheless, Knopp's original proof can be used to prove the corrected inequality.

We'll give two proofs of Knopp's inequality. First, though, we'll take a short detour to prove the integral version of Jensen's inequality, which is not only of interest in its own right, but which is also needed for Knopp's proof.

Jensen's Inequality. *Let f, $w : I \to \mathbb{R}$ be integrable functions with $w(x) \geq 0$ and $\int_I w(x)\,dx = 1$. Let $\Phi : J \to \mathbb{R}$ be a convex function defined on an interval J containing the range of f. Then*

$$\Phi\left(\int_I f(x)\,w(x)\,dx\right) \;\leq\; \int_I \Phi(f(x))\,w(x)\,dx.$$

Proof. Let $\mu = \int_I f(x)\,w(x)\,dx$ and let $T(x) = \Phi(\mu) + m(x-\mu)$ be a supporting line to the graph of Φ at μ. (Recall that we have $T(x) \leq \Phi(x)$ for all x and $T(\mu) = \Phi(\mu)$.) Then $\Phi(\mu) - \Phi(f(x)) \leq m(\mu - f(x))$ for all $x \in I$. Multiplying by w and integrating over I then leads to: $\Phi(\mu) - \int_I \Phi(f(x))\,w(x)\,dx \leq m(\mu-\mu) = 0$. $\square$

If we set $\Phi(x) = e^x$, we have the following integral analogue of the arithmetic-geometric mean inequality:

Corollary. *Let f, $w : I \to (0,\infty)$ with $\int_I w(x)\,dx = 1$. Then*

$$\exp\left(\int_I \log(f(x))\,w(x)\,dx\right) \;\leq\; \int_I f(x)\,w(x)\,dx.$$

A special case of Jensen's inequality is worth stating separately:

Corollary. *Let $f : [\,a,b\,] \to \mathbb{R}$ be continuous and let $\Phi : J \to \mathbb{R}$ be a convex function defined on an interval J containing the range of f. Then*

$$\Phi\left(\frac{1}{b-a}\int_a^b f(t)\,dt\right) \;\leq\; \frac{1}{b-a}\int_a^b \Phi\big(f(t)\big)\,dt.$$

If Φ is strictly convex, then equality can only occur if f is constant.

Proof. We only need to prove the last assertion. As before, let $\mu = (b-a)^{-1} \int_a^b f(t)\,dt$ and let $T(x) = \Phi(\mu) + m(x-\mu)$ be a supporting line for Φ at μ. If Φ is strictly convex, note that $\Phi(y) - \Phi(\mu) > m(y-\mu)$ for $y \neq \mu$. Now, if f is not constant, we have $f(x) \neq \mu$ for all x in some subinterval I of $[\,a, b\,]$. Thus $\Phi(f(x)) - \Phi(\mu) > m(f(x)-\mu)$ on I and $\Phi(f(x)) - \Phi(\mu) \geq m(f(x)-\mu)$ in any case. It follows that $\int_a^b \Big(\Phi(f(x)) - \Phi(\mu)\Big)\,dx > 0$. Conversely, $\int_a^b \Big(\Phi(f(x)) - \Phi(\mu)\Big)\,dx = 0$ would mean that $\Phi(f(x)) = \Phi(\mu)$ for all x. But strictly convex functions can take on the same value at most twice; thus, $f(x)$ assumes at most two values. As f is continuous, this forces f to be constant. $\square$

As a second application of Jensen's inequality, we'll revisit an inequality we saw last week; namely,

$$\int_m^{m+1}\int_n^{n+1} \frac{dy\,dx}{x+y} \;>\; \frac{1}{m+n+1}.$$

To see how this follows from Jensen's inequality, again note that the function $h(x) = 1/(c+x)$ is strictly convex for $x > -c$. Thus,

$$\int_n^{n+1} \frac{dy}{x+y} \;>\; \frac{1}{x + \int_n^{n+1} y\,dy} \;=\; \frac{1}{x+n+1/2}.$$

Similarly,

$$\int_m^{m+1} \frac{dx}{x+n+1/2} \;>\; \frac{1}{n+1/2+\int_m^{m+1} x\,dx} \;=\; \frac{1}{m+n+1}.$$

One more easy calculation and we'll be ready to prove Knopp's inequality.

Lemma. *Let $f : [\,0,\infty) \to (0,\infty)$ be integrable and define $g(x) = \dfrac{1}{x^2}\displaystyle\int_0^x yf(y)\,dy$. Then*

$$\int_0^\infty f(x)\,dx = \int_0^\infty g(x)\,dx.$$

Proof. This is a simple matter of changing the order of integration.

$$\int_0^\infty g(x)\,dx \;=\; \int_0^\infty \frac{1}{x^2}\int_0^x yf(y)\,dy\,dx \;=\; \int_0^\infty yf(y)\int_y^\infty \frac{1}{x^2}\,dx\,dy \;=\; \int_0^\infty f(y)\,dy. \quad \square$$

Proof of Knopp's inequality. Recall that an antiderivative for $\log x$ is $x\log x - x$ (which vanishes at 0). Thus,

$$\begin{aligned}
\exp\left(\frac{1}{x}\int_0^x \log(f(y))\,dy\right) &= \exp\left(\frac{1}{x}\int_0^x \log(yf(y))\,dy - \frac{1}{x}\int_0^x \log y\,dy\right) \\
&= \frac{e}{x}\cdot\exp\left(\frac{1}{x}\int_0^x \log(yf(y))\,dy\right) \\
&\le \frac{e}{x}\cdot\frac{1}{x}\int_0^x \exp\big(\log(yf(y)\big)\,dy \qquad (2)\\
&= \frac{e}{x^2}\int_0^x yf(y)\,dy,
\end{aligned}$$

where, in (2), we've applied Jensen's inequality. An appeal to our previous Lemma yields

$$\int_0^\infty \exp\left\{\frac{1}{x}\int_0^x \log\big(f(t)\big)\,dt\right\}dx \;\le\; e\int_0^\infty f(x)\,dx. \tag{3}$$

Now equality in (3) would force equality in (2) for all x, which would in turn force $\log(yf(y))$ to be constant. If f is not identically zero, this would mean that $f(y) = c/y$ for some positive constant c. This is clearly inconsistent with the fact that f is integrable over $(0,\infty)$. Thus, we have strict inequality in (3) unless $f = 0$. $\square$

We next present an inequality due to Lennart Carleson (1954) which is at least partly related to Jensen's inequality, through its style of proof, and which is at least partly related to Hardy's inequality, in a way that will become apparent later. Moreover, it will yield both Carleman's inequality and Knopp's inequality as corollaries.

Carleson's Inequality. *Let $\varphi : [0,\infty) \to \mathbb{R}$ be a differentiable convex function with $\varphi(0) = 0$. Then, for any $-1 < \alpha < \infty$, we have*

$$\int_0^\infty x^\alpha \exp\left(-\frac{\varphi(x)}{x}\right)dx \;\le\; e^{\alpha+1}\int_0^\infty x^\alpha \exp(-\varphi'(x))\,dx. \tag{4}$$

The constant $e^{\alpha+1}$ is best possible.

Proof. To begin, note that if $p > 1$, then the convexity of φ assures us that

$$\frac{\varphi(py) - \varphi(y)}{py - y} \ge \varphi'(y), \tag{5}$$

which we will write as: $-\varphi(py) \le -\varphi(y) - (p-1)y\varphi'(y)$. Now we estimate the left-hand side of (4) using (5) and Hölder's inequality with $q = p/(p-1)$, the conjugate to p.

$$
\begin{aligned}
I_A = \int_0^A x^\alpha \exp\left(-\frac{\varphi(x)}{x}\right) dx = p^{\alpha+1} &\int_0^{A/p} y^\alpha \exp\left(-\frac{\varphi(py)}{py}\right) dy \\
\le p^{\alpha+1} &\int_0^A y^\alpha \exp\left(\frac{-\varphi(y) - (p-1)y\varphi'(y)}{py}\right) dy \\
= p^{\alpha+1} &\int_0^A y^{\alpha/p} \exp\left(\frac{-\varphi(y)}{py}\right) \cdot y^{\alpha/q} \exp\left(-\frac{\varphi'(y)}{q}\right) dy \\
\le p^{\alpha+1}\, &I_A^{1/p} \left\{ \int_0^A y^\alpha \exp(-\varphi'(y))\, dy \right\}^{1/q}.
\end{aligned}
$$

We now divide by $I_A^{1/p}$ and raise both sides to the q-th power to arrive at

$$
\int_0^A x^\alpha \exp\left(-\frac{\varphi(x)}{x}\right) dx \;\le\; \left(p^{\,p/(p-1)}\right)^{\alpha+1} \int_0^A x^\alpha \exp(-\varphi'(x))\, dx.
$$

To finish the proof, we first let $A \to \infty$ and then let $p \to 1^+$ or, equivalently, let $q \to \infty$. Written in terms of q we have

$$
p^{\,p/(p-1)} = \left(\frac{q}{q-1}\right)^q = \left(1 + \frac{1}{q-1}\right)^q \to e \quad \text{as} \quad q \to \infty. \qquad \square
$$

As an application of Carleson's inequality, let's see how it can be used to derive Carleman's inequality.

First suppose that the sequence (a_n) is nonnegative and *decreasing* and define $s_0 = 0$, $s_n = \sum_{k=1}^n \log(1/a_k)$. Now let φ be the piecewise linear continuous function with "nodes" or "corners" at $(n, \varphi(n)) = (n, s_n)$. That is $\varphi(n) = s_n$ for each n and φ is defined linearly on each interval $(n-1, n)$. Then, on the open interval $(n-1, n)$, we'll have $\varphi'(x) = s_n - s_{n-1} = \log(1/a_n)$. But if (a_n) decreases, then φ' will increase; thus, φ will be convex. Also, because $\varphi(0) = 0$, the function $\varphi(x)/x$ will increase—it's the slope of the chord from $(0,0)$ to $(x, \varphi(x))$. Moreover, note that for $n-1 < x < n$ we have

$$
\frac{\varphi(x)}{x} \le \frac{\varphi(n)}{n} = \frac{1}{n} \sum_{k=1}^n \log(1/a_k). \tag{6}
$$

Equality in (6) for all n and x would force (a_k) to be *constant*, which is plainly inconsistent with the summability of (a_k). Thus, we must have strict inequality in (6), at least for certain n and x (which, by the continuity of φ, will be good enough for our purposes).

Finally, notice that

$$\left(\prod_{k=1}^{n} a_k\right)^{1/n} = \exp\left(-\frac{\varphi(n)}{n}\right) \leq \int_{n-1}^{n} \exp\left(-\frac{\varphi(x)}{x}\right) dx,$$

where the inequality is strict, at least for certain values of n, per our discussion of (6). Summing this inequality and applying Carleson's inequality leads to

$$\sum_{n=1}^{\infty}\left(\prod_{k=1}^{n} a_k\right)^{1/n} \;<\; \int_0^{\infty} \exp\left(-\frac{\varphi(x)}{x}\right) dx \;\leq\; e\int_0^{\infty} \exp\left(-\varphi'(x)\right) dx \;=\; e\sum_{n=1}^{\infty} a_n,$$

where, again, strict inequality follows from our remarks concerning (6). That is, we have Carleman's inequality for *decreasing* sequences (a_n). But that hardly matters:

Claim: If (a_k) is decreasing and if (b_k) is any rearrangement of (a_k), then $b_1 b_2 \cdots b_n \leq a_1 a_2 \cdots a_n$ for all n.

Indeed, if n is *fixed*, we may suppose that $b_1, b_2, \ldots, b_n$ have been arranged in decreasing order, in which case it's clear that $b_1 \leq a_1$, $b_2 \leq a_2$, and so on. Thus, $b_1 b_2 \cdots b_n \leq a_1 a_2 \cdots a_n$. It follows that

$$\sum_{n=1}^{\infty}\left(\prod_{k=1}^{n} b_k\right)^{1/n} \;\leq\; \sum_{n=1}^{\infty}\left(\prod_{k=1}^{n} a_k\right)^{1/n} \;<\; e\sum_{n=1}^{\infty} a_n \;=\; e\sum_{n=1}^{\infty} b_n.$$

(Because $a_n \geq 0$, the series is unconditionally summable—that is, every rearrangement of (a_n) will sum to the same value.)

Carleson's inequality also leads to a version of Knopp's inequality valid for *decreasing* functions. In this case, the function $\varphi(x) = -\int_0^x \log(f(t))\,dt$ will have $\varphi'(x) = -\log(f(x))$, which is increasing; hence, φ is convex.

Corollary. *If* $f : [0,\infty) \to (0,\infty)$ *is* decreasing, *then*

$$\int_0^{\infty} \exp\left\{\frac{1}{x}\int_0^{x} \log\bigl(f(t)\bigr)\,dt\right\} dx \;\leq\; e\int_0^{\infty} f(x)\,dx.$$

In light of our discussion of decreasing sequences and our knowledge of Knopp's result, this Corollary raises a natural question: If f is decreasing, and if g is a "rearrangement" of f (whatever that might mean), does it follow that

$$\int_0^\infty \exp\left\{\frac{1}{x}\int_0^x \log\bigl(g(t)\bigr)\,dt\right\} dx \;\leq\; \int_0^\infty \exp\left\{\frac{1}{x}\int_0^x \log\bigl(f(t)\bigr)\,dt\right\} dx\ ?$$

1. (a) Given positive weights $w_1, \ldots, w_n$, show that

$$\left(\sum_{k=1}^{n} a_k\right)^2 \le \left(\sum_{k=1}^{n} \frac{1}{w_k}\right)\left(\sum_{k=1}^{n} a_k^2 w_k\right).$$

(b) If we set $w_k = w_k(t) = t + k^2/t$ for $t > 0$, show that the first factor on the right is bounded above by $\pi/2$.

(c) Show that: $\min\left\{\sum_{k=1}^{n} a_k^2 w_k(t) : t > 0\right\} = 2\left(\sum_{k=1}^{n} a_k^2\right)^{1/2}\left(\sum_{k=1}^{n} k^2 a_k^2\right)^{1/2}$.

(d) Finally, conclude that $\left(\sum_{k=1}^{n} a_k\right)^4 \le \pi^2\left(\sum_{k=1}^{n} a_k^2\right)\left(\sum_{k=1}^{n} k^2 a_k^2\right)$. This is known as Carlson's inequality. The constant π^2 is best possible.

2. For any pair of nonnegative, real sequences (a_m) and (b_n), show that

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{a_m b_n}{\max\{m,n\}} < 4\left(\sum_{m=1}^{\infty} a_m^2\right)^{1/2}\left(\sum_{n=1}^{\infty} b_n^2\right)^{1/2}.$$

[Hint: Mimic either of the proofs we gave for Hilbert's inequality (with $p = 2$). In the case of the "homogeneous kernel" approach, take $K(x,y) = [\max\{x,y\}]^{-1}$ and note that $\int_0^\infty \left[\sqrt{u}\max\{1,u\}\right]^{-1} du = 4$.]

3. Let (a_n) be a nonnegative sequence and let (b_n) denote a rearrangement of (a_n). If (b_n) is *decreasing*, show that $\sum_{n=1}^{\infty}\left(\frac{A_n}{n}\right)^p \le \sum_{n=1}^{\infty}\left(\frac{B_n}{n}\right)^p$, where $A_n = a_1 + a_2 + \cdots + a_n$ and $B_n = b_1 + b_2 + \cdots + b_n$. [Hint: Argue, for example, that the left-hand side increases if a pair of "out of order" terms $a_i < a_j$, where $i < j$, are swapped.] Conclude that it suffices to prove Hardy's inequality (for series) in the case where (a_n) is decreasing.

4. Let (a_n) be nonnegative and decreasing, and define $f(x) = a_n$ for $n - 1 \le x < n$. Then $\sum_{n=1}^{\infty} a_n^p = \int_0^\infty f(t)^p\,dt$. As usual, we define $F(x) = \int_0^x f(t)\,dt$.

(i) Show that $F(x)/x$ decreases from A_n/n to $A_{n+1}/(n+1)$ over $n < x < n+1$ and conclude that $\int_0^\infty (F(x)/x)^p dx \ge \sum_{n=1}^{\infty}(A_n/n)^p$.

(ii) Deduce that Hardy's series inequality for (a_n) follows from his integral inequality for f. Thus, from **3**, the integral inequality implies the series inequality for any nonnegative (not necessarily decreasing) sequence.

5. Here is an outline of an alternate proof of Hardy's integral inequality due to Ingham. In what follows, all functions are nonnegative and $1 < p < \infty$. Parts (a)–(c) and (e) are independent, only part (d) depends on the prior steps.

(a) Show that $\left(\int_0^1 \left(\int_0^1 g(x,y)\,dx\right)^p dy\right)^{1/p} \le \int_0^1 \left(\int_0^1 g(x,y)^p\,dy\right)^{1/p} dx$. [Hint: Write $J(x) = \int_0^1 g(x,y)\,dy$. Use Hölder on $I = \int_0^1 J(x)^p\,dx = \int_0^1 \int_0^1 J(x)^{p-1} g(x,y)\,dy\,dx$. For a stronger result, show that the inequality is strict unless $g(x,y) = h(x)k(y)$.]

(b) If $F(x) = \int_0^x f(t)\,dt$, show that $F(y)/y = \int_0^1 f(xy)\,dx$.

(c) Show that $\left(\int_0^1 f(xy)^p\,dy\right)^{1/p} \le \left(x^{-1}\int_0^1 f(t)^p\,dt\right)^{1/p}$.

(d) Show that $\left(\int_0^1 (F(y)/y)^p\,dy\right)^{1/p} \le \frac{p}{p-1}\left(\int_0^1 f(t)^p\,dt\right)^{1/p}$.

(e) This proves Hardy's inequality over $[\,0,1\,]$. It's possible to deduce the inequality over $[\,0,\infty)$ from (d). How? [Hint: First try to deduce the result over $[\,0,b\,]$ by means of a simple change of variable.]

6. Here is an outline of Pólya's proof of Carleman's inequality; it's legendary for its elegance and acuity. Let (a_n) be a sequence of positive real numbers.

(a) Given a positive sequence (c_n), justify the steps in the following calculation:

$$\begin{aligned}\sum_{n=1}^{\infty}(a_1a_2\cdots a_n)^{1/n} &= \sum_{n=1}^{\infty}\left(\frac{c_1a_1\cdot c_1a_2\cdots c_na_n}{c_1c_1\cdots c_n}\right)^{1/n}\\ &\le \sum_{n=1}^{\infty}(c_1c_2\cdots c_n)^{-1/n}\cdot\frac{1}{n}\sum_{m=1}^{n}c_ma_m\\ &= \sum_{m=1}^{\infty}a_mc_m\sum_{n=m}^{\infty}\frac{1}{n}(c_1c_2\cdots c_n)^{-1/n}.\end{aligned}$$

(b) If $c_m = (m+1)^m/m^{m-1}$, show that $(c_1c_2\cdots c_n)^{-1/n} = 1/(n+1)$ and conclude that

$$\sum_{n=m}^{\infty}\frac{1}{n}(c_1c_2\cdots c_n)^{-1/n} = \sum_{n=m}^{\infty}\frac{1}{n(n+1)} = \frac{1}{m}.$$

(c) Finally, deduce that

$$\sum_{n=1}^{\infty}(a_1a_2\cdots a_n)^{1/n} \le \sum_{m=1}^{\infty}\frac{a_mc_m}{m} = \sum_{m=1}^{\infty}a_m\left(1+\frac{1}{m}\right)^m < e\sum_{m=1}^{\infty}a_m.$$

7. Show that the constant $e^{\alpha+1}$ in Carleson's inequality is best possible. [Hint: Let $a > \alpha$ and define $\varphi(x) = (a+1)x\log x$ for $x > 1$, $\varphi(x) = 0$ otherwise.]

1. (a) Given positive weights $w_1, \ldots, w_n$, show that

$$\left(\sum_{k=1}^{n} a_k\right)^2 \;\le\; \left(\sum_{k=1}^{n} \frac{1}{w_k}\right)\left(\sum_{k=1}^{n} a_k^2 w_k\right).$$

(b) If we set $w_k = w_k(t) = t + k^2/t$ for $t > 0$, show that the first factor on the right is bounded above by $\pi/2$.

(c) Show that: $\min\left\{\sum_{k=1}^{n} a_k^2 w_k(t) : t > 0\right\} = 2\left(\sum_{k=1}^{n} a_k^2\right)^{1/2}\left(\sum_{k=1}^{n} k^2 a_k^2\right)^{1/2}.$

(d) Finally, conclude that $\left(\sum_{k=1}^{n} a_k\right)^4 \le \pi^2 \left(\sum_{k=1}^{n} a_k^2\right)\left(\sum_{k=1}^{n} k^2 a_k^2\right).$

Solution. (a) This is a straight application of Cauchy-Schwarz:

$$\sum_{k=1}^{n} a_k \;=\; \sum_{k=1}^{n} \frac{1}{\sqrt{w_k}}\, a_k \sqrt{w_k} \;\le\; \left(\sum_{k=1}^{n} \frac{1}{w_k}\right)^{1/2}\left(\sum_{k=1}^{n} a_k^2 w_k\right)^{1/2}.$$

(b) For $w_k = w_k(t) = t + k^2/t$ we have, by the integral test,

$$\sum_{k=1}^{n} \frac{1}{w_k} \;=\; \sum_{k=1}^{n} \frac{t}{t^2+k^2} \;<\; \int_0^\infty \frac{t}{t^2+x^2}\,dx \;=\; \int_0^\infty \frac{1}{1+x^2}\,dx \;=\; \frac{\pi}{2}.$$

(c) Note that $\sum_{k=1}^{n} a_k^2 w_k(t) \;=\; t\sum_{k=1}^{n} a_k^2 + \frac{1}{t}\sum_{k=1}^{n} k^2 a_k^2 \;=\; At + \frac{B}{t}$, which is minimized when $t = (B/A)^{1/2}$ (and its minimum value is then $2A^{1/2}B^{1/2}$).

(d) Finally, we assemble the pieces. From (b) we have $\left(\sum_{k=1}^{n} a_k\right)^2 \;\le\; \frac{\pi}{2}\sum_{k=1}^{n} a_k^2 w_k(t)$ for all $t > 0$. Thus, from (c),

$$\left(\sum_{k=1}^{n} a_k\right)^2 \;\le\; \frac{\pi}{2}\cdot 2\left(\sum_{k=1}^{n} a_k^2\right)^{1/2}\left(\sum_{k=1}^{n} k^2 a_k^2\right)^{1/2}.$$

Squaring both sides completes the solution (except for the claim that the constant π^2 is best possible). $\square$

Majorization and Schur Convexity

In this section we will discuss an order relation that may help to explain—and consolidate—many of the elementary inequalities that we've encountered. As we'll see, *majorization*, as the study and application of this order relation is called, is a fruitful undertaking.

Before we say more, let's make a few definitions and look at a few easy examples. To begin, given a finite length sequence of real numbers $x = (x_1, x_2, \ldots, x_n)$, we will denote the *decreasing rearrangement* of x by $x^* = (x_k^*)$. That is, $x_1^* = \max\{x_k : 1 \le k \le n\} = x_{k_1}$, $x_2^* = \max\{x_k : 1 \le k \le n,\ k \ne k_1\} = x_{k_2}$, and so on. It would be more accurate to say that x^* is the non-increasing rearrangement of x, but that's rather awkward. We'll stick with decreasing.

Here's an easy example of the decreasing rearrangement in action:

Proposition. *For $x,\, y \in \mathbb{R}^n$ we have* $\displaystyle\sum_{i=1}^{n} x_i^* y_{n+1-i}^* \ \le\ \sum_{i=1}^{n} x_i y_i \ \le\ \sum_{i=1}^{n} x_i^* y_i^*.$

Proof. We only need to prove the second inequality; the first then follows by considering x and $-y$. To begin, we may suppose that one of the sequences, let's say x, is already in decreasing order but y is not. Thus, there are a pair of indices $j < k$ such that $y_j < y_k$ while $x_j \ge x_k$. Now consider:

$$x_j y_k + x_k y_j - (x_j y_j + x_k y_k) = (x_k - x_j)(y_j - y_k) \ge 0.$$

It follows that the sum $\sum_{i=1}^n x_i y_i$ can only increase if we exchange y_j and y_k. After a finite number of such exchanges, y would be in decreasing order. $\square$

We next define an order relation, written $x \prec y$, read "x is *majorized* by y" (or "y majorizes x"), and defined by the following string of inequalities (plus one equation):

$$(1)\qquad \begin{cases} x_1^* \le y_1^* \\ x_1^* + x_2^* \le y_1^* + y_2^* \\ \quad\vdots \\ x_1^* + \cdots + x_{n-1}^* \le y_1^* + \cdots + y_{n-1}^* \\ x_1^* + x_2^* + \cdots + x_n^* = y_1^* + y_2^* + \cdots + y_n^* \end{cases}$$

Example. $(1,1,1,1) \prec (0,1,2,1) \prec (2,0,2,0) \prec (1,3,0,0) \prec (0,0,4,0)$.

Judging by this single example, it would seem that "smaller" sequences are more evenly distributed, while "bigger" sequences are less evenly distributed. It's also reasonably clear that "$\prec$" is transitive; thus, $x \prec y$ and $y \prec z$ imply that $x \prec z$. But, because the relation doesn't depend on the order of the terms, we can't expect to have, say, $x \prec y$ and $y \prec x$ imply that $x = y$. It would, however, imply that $x^* = y^*$. Also, because we have one equation to satisfy, not all pairs of sequences will be comparable; for example, $(1,1,1)$ and $(1,0,1)$ are incomparable. Finally, given any x and any permutations σ and τ of $\{1,2,\ldots,n\}$, the *rearrangements* $x_\sigma = (x_{\sigma(1)},\ldots,x_{\sigma(n)})$ and x_τ satisfy $x_\sigma \prec x \prec x_\tau$.

In 1923, Schur studied majorization and asked which functions $f : (\mathbb{R}^n)^+ \to \mathbb{R}$ satisfy $f(x) \le f(y)$ whenever $x \prec y$. We now call such functions *Schur convex.* If, instead, $f(x) \ge f(y)$ whenever $x \prec y$, we say that f is *Schur concave.* Again, more precise labels might be "Schur increasing" or "Schur monotone," but so it goes. Because we have $x_\sigma \prec x \prec x_\tau$ for any permutations σ and τ, note that a Schur convex/concave function must also be *symmetric*; that is, $f(x_\sigma) = f(x)$ for any permutation σ.

Fortunately, Schur gave us more than a label for such functions; he also gave us a test for them.

Schur's Criterion. *Let $f : (\mathbb{R}^n)^+ \to \mathbb{R}$ be symmetric and continuously differentiable. Then f is Schur convex if and only if, for all $1 \le j,\, k \le n$ and all $x \in (\mathbb{R}^n)^+$,*

$$0 \le (x_j - x_k)\left(\frac{\partial f}{\partial x_j} - \frac{\partial f}{\partial x_k}\right).$$

Proof. Because f is symmetric, it follows that f is Schur convex on $(\mathbb{R}^n)^+$ if and only if f is Schur convex on $\mathcal{D} = \{x : x_1 \ge x_2 \ge \cdots \ge x_n \ge 0\}$.

Given $x \in (\mathbb{R}^n)^+$, let's agree to write $\tilde{x} = (\tilde{x}_1,\ldots,\tilde{x}_n)$ for the vector with entries $\tilde{x}_k = x_1 + \cdots + x_k$, $1 \le k \le n$. With this notation, the majorization $x \prec y$ is equivalent to

the string of inequalities $\tilde{x}_k \le \tilde{y}_k$, $1 \le k < n$, together with the equation $\tilde{x}_n = \tilde{y}_n$. In this way, majorization is almost the same as the coordinatewise comparison $\tilde{x} \le \tilde{y}$.

In particular, if we put $\widetilde{\mathcal{D}} = \{\tilde{x} : x \in \mathcal{D}\}$ and define $\tilde{f} : \widetilde{\mathcal{D}} \to \mathbb{R}$ by $\tilde{f}(\tilde{x}) = f(x)$, then f is Schur convex on $\mathcal{D}$ if and only if $\tilde{f}$ satisfies $\tilde{f}(\tilde{x}) \le \tilde{f}(\tilde{y})$ whenever $\tilde{x}$, $\tilde{y} \in \widetilde{\mathcal{D}}$ satisfy $\tilde{x}_j \le \tilde{y}_j$, $1 \le j < n$, and $\tilde{x}_n = \tilde{y}_n$. That is, $\tilde{f}$ must be nondecreasing in each of its first $n-1$ coordinates; hence,

$$0 \le \frac{\partial \tilde{f}(\tilde{x})}{\partial \tilde{x}_j} \quad \text{for all} \quad 1 \le j < n.$$

But $\tilde{f}(\tilde{x}) = f(\tilde{x}_1, \tilde{x}_2 - \tilde{x}_1, \ldots, \tilde{x}_n - \tilde{x}_{n-1})$ so, by the chain rule,

$$0 \;\le\; \frac{\partial \tilde{f}(\tilde{x})}{\partial \tilde{x}_j} \;=\; \frac{\partial f(x)}{\partial x_j}\frac{\partial x_j}{\partial \tilde{x}_j} + \frac{\partial f(x)}{\partial x_{j+1}}\frac{\partial x_{j+1}}{\partial \tilde{x}_{j+1}} \;=\; \frac{\partial f(x)}{\partial x_j} - \frac{\partial f(x)}{\partial x_{j+1}} \tag{2}$$

for all $1 \le j < n$. Summing (2) over $j, j+1, \ldots, k-1$ gives us

$$0 \le \frac{\partial f(x)}{\partial x_j} - \frac{\partial f(x)}{\partial x_k} \quad \text{for} \quad 1 \le j < k \le n \quad \text{and} \quad x \in \mathcal{D}.$$

Because f is symmetric, this is equivalent to

$$0 \le (x_j - x_k)\left(\frac{\partial f(x)}{\partial x_j} - \frac{\partial f(x)}{\partial x_k}\right) \quad \text{for} \quad x \in (\mathbb{R}^n)^+,$$

which is just a way of saying that the the variables must be in the same order as the corresponding partial derivatives. $\square$

By way of an example, let's check that $f(x) = x_1 x_2 \ldots x_n$ is Schur concave on $(\mathbb{R}^n)^+$. In this case, $f_{x_j} = \prod_{i \ne j} x_i$, and so

$$(x_j - x_k)(f_{x_j} - f_{x_k}) \;=\; -(x_j - x_k)^2 \prod_{i \ne j,k} x_i \;\le\; 0.$$

It follows that $f(x) \le f(\bar{x})$, where $\bar{x}$ is the constant vector with entries $M_1(x) = (x_1 + \cdots + x_n)/n$, because $\bar{x} \prec x$. That is, we have another proof of the AGM: $x_1 x_2 \cdots x_n \le M_1(x)^n$.

As another example, it's not hard to check that $f(x) = \|x\|_p$ is Schur convex for $1 \le p < \infty$ (and Schur concave for $0 < p < 1$). Thus, using the same "test vector" as above, we have $f(\bar{x}) \le f(x)$ or: $M_1(x) \le M_p(x)$. It also tells us that the vectors in our first

example are, in fact, listed in increasing order of p-norm. In this case, $4^{1/p} \leq (2+2^p)^{1/p} \leq (2^p + 2^p)^{1/p} \leq (1+3^p)^{1/p} \leq 4$.

Rather than catalogue more examples, let's turn our attention to a better understanding of the relation $x \prec y$. This will lead us to another test for Schur convexity.

What's missing is some justification for the "convexity" in Schur convexity and, ultimately, its connection with majorization. Filling in the details will take some time, and will take us on something of a side trip, but the journey will have its rewards.

A Bit of Matrix Theory

We begin with the notion of a *permutation matrix.* Given a permutation σ of $\{1, \ldots, n\}$, the matrix associated with the transformation $T_\sigma(x) = x_\sigma$ is called a permutation matrix. Note that T_σ permutes the basis vectors according to σ; i.e., $T_\sigma(e_j) = e_{\sigma(j)}$, and, hence, its matrix representation is the corresponding permutation of the columns of I, the identity matrix. Said in other words, a permutation matrix is a matrix with entries 0 and 1 in which every row and every column has precisely one nonzero entry. The simplest example of a permuation matrix is a *transposition matrix*, which corresponds to a simple exchange, or transposition, of two basis vectors:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \longrightarrow \quad \text{swap } e_2 \text{ and } e_4 \quad \longrightarrow \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

It won't surprise you to learn that the set of permutation matrices on $\mathbb{R}^n$ forms a group (under multiplication) that is isomorphic to S_n, the symmetric group on n letters (that is, the group of all permutations on $\{1, \ldots, n\}$). In particular, note that every permuation matrix is invertible and its inverse is again a permutation matrix; indeed, $(T_\sigma)^{-1} = T_{\sigma^{-1}}$.

We will also need the notion of a *transfer matrix*, defined by $T = (1-\lambda)I + \lambda T_\sigma$, where $0 \leq \lambda \leq 1$ and where T_σ is a transposition matrix. Symbolically, if T_σ exchanges e_i

and e_j, we would have

$$T = \begin{array}{cc}
\begin{bmatrix}
1 & & & & & & \\
& \ddots & & & & & \\
& & 1-\lambda & \cdots & \lambda & & \\
& & \vdots & \ddots & \vdots & & \\
& & \lambda & \cdots & 1-\lambda & & \\
& & & & & \ddots & \\
& & & & & & 1
\end{bmatrix} &
\begin{array}{l} \\ \\ \leftarrow i \\ \\ \leftarrow j \\ \\ \\ \end{array} \\
\begin{array}{ccccccc} & & \uparrow & & \uparrow & & \\ & & i & & j & & \end{array} &
\end{array}$$

where all other diagonal entries are 1 and all other off-diagonal entries are 0. To understand the action of T, suppose (for sake of argument) that we have $x_i > x_j$; then T subtracts a bit of the excess (namely, $x_i - x_j$) from x_i and transfers it to x_j; that is,

$$\text{(3)} \qquad \begin{cases} (T(x))_i = (1-\lambda)x_i + \lambda x_j = x_i - \lambda(x_i - x_j), \\ (T(x))_j = \lambda x_i + (1-\lambda)x_j = x_j + \lambda(x_i - x_j), \\ (T(x))_k = x_k, \quad k \neq i,\, j, \end{cases}$$

where $(T(x))_k$ denotes the k-th coordinate of $T(x)$. Also notice that we have $(T(x))_i + (T(x))_j = x_i + x_j$; thus, x and $T(x)$ are comparable and, in fact, $T(x) \prec x$ because $T(x)$ is more evenly distributed than x (but it also follows from direct calculation, as we'll see shortly). Finally, notice that every transposition matrix (including I) is also a transfer matrix (by taking $\lambda = 0$ or $\lambda = 1$).

More generally, suppose that we're given a convex combination of several permutation matrices:

$$D = \lambda_1 T_{\sigma_1} + \cdots + \lambda_k T_{\sigma_k}, \quad \lambda_i \geq 0, \quad \lambda_1 + \cdots + \lambda_k = 1.$$

Now the permutation matrices T_{σ_i} each have a single 1 in any given row or column, so their convex combination D must satisfy

$$\sum_{i=1}^{n} d_{i,j} = \sum_{i=1}^{n} \lambda_i = 1 \quad \text{and} \quad \sum_{j=1}^{n} d_{i,j} = \sum_{j=1}^{n} \lambda_j = 1$$

for all i and j; that is, every row and every column of D sums to 1. Such a matrix is said to be *doubly stochastic*. In other words, an $n \times n$ matrix is doubly stochastic if $d_{i,j} \geq 0$, for all

i, j, and every row and every column sums to 1; i.e., $\sum_{i=1}^{n} d_{i,j} = 1$ and $\sum_{j=1}^{n} d_{i,j} = 1$ for all i and j. (It follows that $d_{i,j} \leq 1$ for all i, j.) As it happens, every doubly stochastic matrix can be written as a convex combination of permutation matrices (Birkhoff's theorem), so our initial supposition is actually the general case.

Now might be a good time to say a couple of words about permutations and transpositions. From the proof of our first Proposition, it's not hard to see that any permutation can be written as the product of finitely many transpositions. To see this, let's first establish a useful claim:

Claim. Given $y \in \mathbb{R}^n$, we can write $y^* = T_k T_{k-1} \cdots T_1 y$ for some (finite) sequence of transposition matrices.

We've sketched the proof already: If i_1 is the index of the largest coordinate of y and if τ_1 is the transposition that exchanges i_1 with 1, then $T_1 = T_{\tau_1}$ moves the largest entry of y into the first coordinate; that is $T_1 y$ now has its largest entry in the first coordinate. If $i_2 \neq 1$ is the index of the second largest coordinate in $T_1 y$ and if τ_2 exchanges i_2 and 2, then $T_2 = T_{\tau_2}$ moves the second largest entry of $T_1 y$ into the second coordinate. That is, $T_2 T_1 y$ agrees with y^* in its first two coordinates. And so on. After at most $n-1$ such exchanges, we'll arrive at y^*.

Now this argument has very little to do with decreasing rearrangements and everything to do with matching a specific permutation of the coordinates of y, so we've actually established:

Claim. Given $y \in \mathbb{R}^n$ and $\sigma \in S_n$, we can write $y_\sigma = T_k T_{k-1} \cdots T_1 y$ for some (finite) sequence of transposition matrices. In other words, $T_\sigma = T_k T_{k-1} \cdots T_1$ or, in still other words, $\sigma = \tau_k \tau_{k-1} \cdots \tau_1$. That is, every permutation can be written as the product of finitely many transpositions.

We're now ready to put some of the pieces together.

Theorem. *Let $x, y \in \mathbb{R}^n$ with $x, y \geq 0$. Then the following are equivalent*

(i) *x is a convex combination (of finitely many) of the vectors $\{y_\sigma : \sigma \in S_n\}$.*

(ii) *$x = Dy$ for some doubly stochastic matrix D.*

(iii) *$x \prec y$.*

(iv) *$x = T_1 T_2 \cdots T_k y$ for some (finite) sequence of transfer matrices $T_1, \ldots, T_k$.*

Proof. We've actually already shown that (i) implies (ii). Indeed, we only need to recall that $y_\sigma = T_\sigma(y)$ for then:

$$x = \lambda_1 y_{\sigma_1} + \cdots + \lambda_k y_{\sigma_k} = (\lambda_1 T_{\sigma_1} + \cdots + \lambda_k T_{\sigma_k})(y) = Dy$$

where D is doubly stochastic.

To see that (ii) implies (iii), first note that either condition is unaffected by permutations of x or y. As we've already seen, the condition $x \prec y$ is equivalent to the condition $x_\sigma \prec y_\tau$ for any permutations σ, τ. Moreover, because the product of a doubly stochastic matrix and a permutation matrix is still doubly stochastic, it follows that the condition $x = Dy$ is equivalent to the condition $x_\sigma = D' y_\tau$ for any permutations σ, τ (and some doubly stochastic matrix D', which may depend on σ and τ, of course). Thus we may suppose that both x and y are in *decreasing order.*

Now if $x = Dy$ and if $1 \leq k \leq n$ is fixed, then we can write

$$x_i = \sum_{j=1}^n d_{i,j} y_j \implies \sum_{i=1}^k x_i = \sum_{i=1}^k \sum_{j=1}^n d_{i,j} y_j = \sum_{j=1}^n c_j y_j \quad \text{where} \quad c_j = \sum_{i=1}^k d_{i,j}.$$

In particular, if $k = n$, then $c_j = 1$ for all j and, hence, $\sum_{i=1}^n x_i = \sum_{i=1}^n y_i$. For $k < n$, the fact that D is doubly stochastic implies that $0 \leq c_j \leq 1$ for all j and

$$\sum_{j=1}^n c_j = \sum_{i=1}^k \sum_{j=1}^n d_{i,j} = k.$$

Thus, for $1 \le k < n$, we can write (watch closely!):

$$\begin{aligned}
\sum_{i=1}^{k} x_i - \sum_{i=1}^{k} y_i &= \sum_{j=1}^{n} c_j y_j - \sum_{i=1}^{k} y_i \\
&= \sum_{j=1}^{n} c_j y_j - \sum_{i=1}^{k} y_i + y_k \left(k - \sum_{j=1}^{n} c_j \right) \\
&= \sum_{j=1}^{n} c_j (y_j - y_k) + \sum_{i=1}^{k} (y_k - y_i) \\
&= \sum_{i=1}^{k} (y_k - y_i)(1 - c_i) + \sum_{i=k+1}^{n} (y_i - y_k) c_i \ \le \ 0.
\end{aligned}$$

In other words, we've shown that $x \prec y$.

We next show that (iii) implies (iv). Let x, $y \in \mathbb{R}^n$ with $x \prec y$. Again, we may suppose that x and y are decreasing. We may also suppose that $x \neq y$, for otherwise we just take $T_1 = I$ in (iv). Our proof proceeds by induction on the number of coordinates N in which x and y differ. The fact that we have $x_1 + \cdots + x_n = y_1 + \cdots + y_n$ will convince you that x and y cannot differ in only one coordinate, so let's first suppose that x and y differ in precisely 2 coordinates; that is, for some $1 \le j < k \le n$ we have $x_j \neq y_j$, $x_k \neq y_k$ and $x_i = y_i$ for $i \neq j$, k. Now, because $x_1 + \cdots + x_j \le y_1 + \cdots + y_j = x_1 + \cdots + x_{j-1} + y_j$, we must have $x_j < y_j$. It follows that we then have $x_k > y_k$ to compensate. Thus, $y_j > x_j \ge x_k > y_k$. Finally, notice that $x_j + x_k = y_j + y_k$; that is, $y_j - x_j = x_k - y_k$.

With (3) as our guide, we can now write the transfer matrix that maps y into x: We set $T = (1 - \lambda)I + \lambda T_{(j,k)}$ where $\lambda = (y_j - x_j)/(y_j - y_k) = (x_k - y_k)/(y_j - y_k)$ and where $T_{(j,k)}$ is the transposition matrix that exchanges e_j and e_k. As the following 2×2 calculation shows, we'll then have $x = Ty$ because:

$$\begin{bmatrix} 1-\lambda & \lambda \\ \lambda & 1-\lambda \end{bmatrix} \begin{bmatrix} y_j \\ y_k \end{bmatrix} = \begin{bmatrix} y_j - \lambda(y_j - y_k) \\ y_k + \lambda(y_j - y_k) \end{bmatrix} = \begin{bmatrix} x_j \\ x_k \end{bmatrix}$$

(and because T fixes all other entries of y). This proves the case $N = 2$.

Suppose now that (iii) implies (iv) holds whenever $x \prec y$ and x and y differ in $N-1$ or fewer coordinates and that we're given a pair of (decreasing) sequences with $x \prec y$ where

x and y differ in N coordinates. Then, as in the case $N = 2$, the first place where x and y differ will have to satisfy $x_j < y_j$ and, for some $k > j$ we'll have $x_k > y_k$ to compensate. But, in fact, we must have some pair of coordinates $j < k$ which satisfy:

$$x_j < y_j, \quad x_k > y_k, \quad \text{and} \quad x_i = y_i \quad \text{for} \quad j < i < k.$$

(We might have $k = j + 1$, of course; what matters here is that x and y do not differ in any intermediate coordinates.)

Because we've assumed that x and y are in decreasing order, this again means that $y_j > x_j \geq x_k > y_k$. In particular, $y_j - y_k > x_j - x_k$ and we'll transfer a bit of y_j to y_k to arrive at x. However, because x and y might differ in several coordinates, we can no longer be assured that $y_j - x_j = x_k - y_k$; instead, we'll transfer (a portion of) the smaller of the two amounts. Specifically, set $T = (1 - \lambda)I + \lambda T_{(j,k)}$ where $T_{(j,k)}$ is the transposition matrix that exchanges e_j and e_k, and where $\lambda = \min\{y_j - x_j, x_k - y_k\}/(y_j - y_k)$. If we let $\tilde{y} = Ty$, then, as above, T acts on only two coordinates of y; in this case,

$$\begin{bmatrix} 1-\lambda & \lambda \\ \lambda & 1-\lambda \end{bmatrix} \begin{bmatrix} y_j \\ y_k \end{bmatrix} = \begin{bmatrix} y_j - \lambda(y_j - y_k) \\ y_k + \lambda(y_j - y_k) \end{bmatrix} = \begin{bmatrix} y_j - \min\{y_j - x_j, x_k - y_k\} \\ y_k + \min\{y_j - x_j, x_k - y_k\} \end{bmatrix} = \begin{bmatrix} \tilde{y}_j \\ \tilde{y}_k \end{bmatrix}$$

and $\tilde{y}_i = y_i$ for $i \neq j$, k. It's reasonably clear from the definition of $\tilde{y}$ that we have

$$y_j \geq \tilde{y}_j \geq x_j \geq x_k \geq \tilde{y}_k \geq y_k.$$

Moreover, because we have either $\lambda = (y_j - x_j)/(y_j - y_k)$ or $\lambda = (x_k - y_k)/(y_j - y_k)$, it follows that we have either $\tilde{y}_j = x_j$ (in the first case) or $\tilde{y}_k = x_k$ (in the second case).

Claim. $x \prec \tilde{y} = Ty \prec y$.

That $\tilde{y} = Ty \prec y$ follows from our proof that (ii) implies (iii). What remains is to show that $x \prec \tilde{y}$. The key to this is the fact that $\tilde{y}_j + \tilde{y}_k = y_j + y_k$. Indeed, we have

$$x_1 + \cdots + x_i \leq y_1 + \cdots + y_i = \tilde{y}_1 + \cdots + \tilde{y}_i \quad \text{if} \quad 1 \leq i < j \quad \text{or} \quad k \leq i \leq n; \tag{4}$$

$$x_1 + \cdots + x_i \leq \tilde{y}_1 + \cdots + \tilde{y}_i \leq y_1 + \cdots + y_i \quad \text{if} \quad j \leq i < k. \tag{5}$$

That (4) holds is immediate; that (5) holds follows from the facts that $x_j \leq \tilde{y}_j$ and $\tilde{y}_i = y_i$ for $j < i < k$.

In summary, we shown that there is a transfer matrix T such that $x \prec \tilde{y} = Ty \prec y$ and such that x and $\tilde{y}$ differ in at most $N-1$ coordinates. By our induction hypothesis, there exist finitely many transfer matrices $T_1, \ldots, T_k$ such that $x = T_1 \cdots T_k(\tilde{y}) = T_1 \cdots T_k(Ty) = (T_1 \cdots T_k T)(y)$. Thus, we've shown that (iii) implies (iv).

Finally, the proof that (iv) implies (i) follows by direct calculation and the observation that if T is a transfer matrix, then $T(y) = (1-\lambda)y + \lambda y_\tau$ is a convex combination of permutations of y. (In other words, if we write H for the set of all convex combinations of the vectors y_τ, then for any transfer matrix T we have $T(H) \subset H$.) $\square$

Corollary. *$\Phi : (\mathbb{R}^n)^+ \to \mathbb{R}$ is Schur convex if and only if $\Phi(T(x)) \leq \Phi(x)$ for every transfer matrix T if and only if $\Phi(D(x)) \leq \Phi(x)$ for every doubly stochastic matrix D.*

Schur's Majorization Inequality. *If $f : I \to \mathbb{R}$ is convex, then the function $\Phi : I^n \to \mathbb{R}$ defined by*

$$\Phi(x_1, \ldots, x_n) = \sum_{i=1}^{n} f(x_i)$$

is Schur convex. That is,

$$\sum_{i=1}^{n} f(x_i) \leq \sum_{i=1}^{n} f(y_i)$$

whenever $x \prec y$.

Proof. The proof is virtually immediate from the preceding Corollary and (3). Indeed, if T is a transfer matrix that acts only on the j-th and k-th coordinates, and if we write $\tilde{x} = T(x)$, then, from (3), we have $\tilde{x}_j = (1-\lambda)x_j + \lambda x_k$, $\tilde{x}_k = \lambda x_j + (1-\lambda)x_k$, and $\tilde{x}_i = x_i$ for $i \neq j$, k. It's then immediate from the convexity of f that $\Phi(T(x)) \leq \Phi(x)$. $\square$

It's clear that if $\Phi(x)$ is Schur convex and if $h(x)$ is increasing, then $h(\Phi(x))$ is again Schur convex. This explains why $f(x) = \|x\|_p$ is Schur convex for $1 \leq p < \infty$. The

function $\Phi(x) = \sum_{i=1}^n x_i^p$ is Schur convex (from the previous Corollary) and $h(x) = x^{1/p}$ is increasing.

By way of another example, we present a classical inequality, due to Muirhead (1903), who introduced the transfer method.

Muirhead's Theorem. *Let $a_1, \ldots, a_n$ be positive. For $x \in \mathbb{R}^n$ with $x \geq 0$ define*

$$\Phi(x) = \sum_{\sigma \in S_n} a_{\sigma(1)}^{x_1} a_{\sigma(2)}^{x_2} \cdots a_{\sigma(n)}^{x_n}.$$

Then Φ is Schur convex.

Proof. Let $T = (1-\lambda)I + \lambda T_\tau$ be a transfer matrix, where $0 \leq \lambda \leq 1$ and where $\tau = \tau_{(i,j)}$ denotes the transposition of i and j, $i \neq j$. Following (3), we can write

$$x_i = m + d, \quad x_j = m - d, \quad (T(x))_i = m + \mu d, \quad (T(x))_j = m - \mu d,$$

where $\mu = 1 - 2\lambda$ satisfies $-1 \leq \mu \leq 1$. Then

$$\begin{aligned}
2\Phi(x) &= [\,\Phi(x) + \Phi(x_\tau)\,] \\
&= \left[\sum_{\sigma \in S_n} a_{\sigma(1)}^{x_1} a_{\sigma(2)}^{x_2} \cdots a_{\sigma(n)}^{x_n} + \sum_{\sigma \in S_n} a_{\sigma(\tau(1))}^{x_1} a_{\sigma(\tau(2))}^{x_2} \cdots a_{\sigma(n)}^{x_n} \right] \\
&= \sum_{\sigma \in S_n} \left(\prod_{k \neq i,\, j} a_{\sigma(k)}^{x_k} \right) \left[a_{\sigma(i)}^{x_i} a_{\sigma(j)}^{x_j} + a_{\sigma(j)}^{x_i} a_{\sigma(i)}^{x_j} \right] \\
&= \sum_{\sigma \in S_n} \left(\prod_{k \neq i,\, j} a_{\sigma(k)}^{x_k} \right) \left[a_{\sigma(i)}^{m+d} a_{\sigma(j)}^{m-d} + a_{\sigma(i)}^{m-d} a_{\sigma(j)}^{m+d} \right].
\end{aligned}$$

Similarly,

$$2\Phi(T(x)) = \sum_{\sigma \in S_n} \left(\prod_{k \neq i,\, j} a_{\sigma(k)}^{x_k} \right) \left[a_{\sigma(i)}^{m+\mu d} a_{\sigma(j)}^{m-\mu d} + a_{\sigma(i)}^{m-\mu d} a_{\sigma(j)}^{m+\mu d} \right].$$

Consequently,

$$2\Phi(x) - 2\Phi(T(x)) = \sum_{\sigma \in S_n} \left(\prod_{k \neq i,\, j} a_{\sigma(k)}^{x_k} \right) \theta(\sigma),$$

where

$$\theta(\sigma) = a_{\sigma(i)}^m a_{\sigma(j)}^m \left[\left(a_{\sigma(i)}^d a_{\sigma(j)}^{-d} + a_{\sigma(i)}^{-d} a_{\sigma(j)}^d \right) - \left(a_{\sigma(i)}^{\mu d} a_{\sigma(j)}^{-\mu d} + a_{\sigma(i)}^{-\mu d} a_{\sigma(j)}^{\mu d} \right) \right]$$

$$= a_{\sigma(i)}^m a_{\sigma(j)}^m \left[(y_\sigma^d + y_\sigma^{-d}) - (y_\sigma^{\mu d} + y_\sigma^{-\mu d}) \right],$$

and where $y_\sigma = a_{\sigma(i)}/a_{\sigma(j)} > 0$. But for $y > 0$, the function $f(s) = y^s + y^{-s}$ is even and increasing on $[0, \infty)$, thus $\theta(\sigma) \geq 0$ because $f(d) \geq f(\mu d)$. Thus, $\Phi(T(x)) \leq \Phi(x)$. □

Now if we insist that $\sum_{i=1}^n x_i = 1$, then the expression

$$\Phi(a, x) = \frac{1}{n!} \sum_{\sigma \in S_n} a_{\sigma(1)}^{x_1} a_{\sigma(2)}^{x_2} \cdots a_{\sigma(n)}^{x_n}$$

defines a mean (of a); it's an average of geometric-like means over the $n!$ members of S_n. With this slight change in notation, we get an interesting extension of the arithmetic-geometric mean inequality:

$$M_0(a) \leq \Phi(a, x) \leq M_1(a)$$

because $b = (1/n, 1/n, \ldots, 1/n) \prec x \prec (1, 0, \ldots, 0) = c$ and because

$$\Phi(a, b) = \frac{1}{n!} \sum_{\sigma \in S_n} (a_{\sigma(1)} a_{\sigma(2)} \cdots a_{\sigma(n)})^{1/n} = (a_1 a_2 \cdots a_n)^{1/n}$$

and

$$\Phi(a, c) = \frac{1}{n!} \sum_{\sigma \in S_n} a_{\sigma(1)} = \frac{1}{n} \sum_{i=1}^n a_i.$$

We complete this section with a proof of Garrett Birkhoff's theorem on doubly stochastic matrices (1946), a pivotal result in the theory of majorization. Curiously, this is yet another example of a famous theorem of uncertain provenance. Some sources claim that it was first proved by König in 1936 (in an article on graph theory). The theorem was discovered independently by John von Neumann in 1953 (in an article on game theory). Many contemporary sources refer to it as the Birkhoff-von Neumann theorem.

Theorem. *Every doubly stochastic matrix can be written as a convex combination of permutation matrices.*

Proof. We'll prove the theorem by induction on the number of nonzero entries. It's clearly true for doubly stochastic matrices with precisely n nonzero entries, as any such matrix

is necessarily a permutation matrix. So we suppose that the theorem holds for doubly stochastic matrices with fewer than k nonzero entries, where $n+1 \leq k \leq n^2$, and that we're given a doubly stochastic matrix D with precisely k nonzero entries.

In particular, D has some entry d_{i_1,j_1} with $0 < d_{i_1,j_1} < 1$. But the sum of entries in the i_1-th row is 1, so there must be some $j_2 \neq j_1$ such that $0 < d_{i_1,j_2} < 1$. But the sum of the entries in the j_2-th column is 1, so there is some index $i_2 \neq i_1$ such that $0 < d_{i_2,j_2} < 1$, and so on. This process can be iterated until some pair (i,j) is repeated. Thus we may suppose that we've chosen a circuit that begins and ends at (i_1, j_1) and, moreover, that we've chosen the shortest such circuit whose entries will then satisfy

$$0 < d_{i_s,j_s} < 1, \quad 0 < d_{i_s,j_{s+1}} < 1, \quad 1 \leq s \leq m,$$

where $i_1, \ldots, i_m$ are distinct, $j_1, \ldots, j_m$ are distinct, and where $j_{m+1} = j_1$. (If the circuit produced three consecutive pairs in the same row or column, we could delete one of those pairs, arriving at a shorter circuit with the property we need. Consequently, a minimal circuit will have an even number of "corners," with no more than two corners in any one row or column.)

We now define an auxiliary $n \times n$ matrix A by setting $a_{i_s,j_s} = 1$, $a_{i_s,j_{s+1}} = -1$, for $1 \leq s \leq m$, and $a_{i,j} = 0$ otherwise. Note that by our construction every row and every column of A sums to zero. Finally, set

$$\lambda = \min_{1 \leq s \leq m} d_{i_s,j_s} > 0 \qquad \text{and} \qquad \mu = \min_{1 \leq s \leq m} d_{i_s,j_{s+1}} > 0.$$

Then each of the matrices $D - \lambda A$ and $D + \mu A$ has nonnegative entries and each has row and column sums equal to 1. Thus, each is doubly stochastic and, moreover, each has fewer than k nonzero entries. By induction, each can be written as a convex combination of permutation matrices. It follows that the same will be true of D because

$$D \; = \; \frac{\mu}{\lambda + \mu}\,(D - \lambda A) + \frac{\lambda}{\lambda + \mu}\,(D + \mu A). \quad \square$$

1. For x, y, $z > 0$, show that

$$\left(\frac{2}{x+y}\right)^5 + \left(\frac{6}{3x+y+2z}\right)^5 + \left(\frac{6}{2x+3y+z}\right)^5 \leq \frac{1}{x^5} + \frac{1}{y^5} + \frac{1}{z^5}.$$

2. For $x \in \mathbb{R}^n$, $n \geq 2$, statisticians use the *sample variance*

$$s(x) = \frac{1}{n-1} \sum_{k=1}^{n} (x_k - \bar{x}\,)^2,$$

where $\bar{x} = (x_1 + \cdots + x_n)/n$, to measure the dispersion of sample data $x = (x_1, \ldots, x_n)$. Information theorists, on the other hand, use the *entropy*

$$h(p) = -\sum_{k=1}^{n} p_k \log p_k$$

to measure the dispersion of probability distributions $p = (p_1, \ldots, p_n)$, where $p_k \geq 0$ and $p_1 + \cdots + p_n = 1$. Show that $s(x)$ is Schur convex on $\mathbb{R}^n$ while $h(p)$ is Schur concave on $(\mathbb{R}^n)^+$.

3. Given $0 < x$, y, $z < 1$ such that $\max\{x, y, z\} \leq (x+y+z)/2$, show that

$$\left(\frac{1+x}{1-x}\right)\left(\frac{1+y}{1-y}\right)\left(\frac{1+z}{1-z}\right) \leq \left\{\frac{1+\frac{1}{2}(x+y+z)}{1-\frac{1}{2}(x+y+z)}\right\}^2.$$

4. Show that every doubly stochastic matrix has a positive diagonal. [Hint: If D is doubly stochastic, then $D = \lambda_1 P_1 + \cdots + \lambda_k P_k$, where each P_j is a permutation matrix, and where some $\lambda_j > 0$.]

5. Let $\lambda_1 \geq \cdots \geq \lambda_n$ be the eigenvalues of a Hermitian matrix $A \in M_n(\mathbb{C})$, arranged in decreasing order. Show that for any orthonormal sequence $v_1, \ldots, v_k \in \mathbb{C}^n$ we have

$$\sum_{j=1}^{k} \lambda_{n-j+1} \;\leq\; \sum_{j=1}^{k} \langle\, A v_j, v_j \,\rangle \;\leq\; \sum_{j=1}^{k} \lambda_j.$$

Conclude that

$$\lambda_n \;\leq\; \inf\{\langle\, Av, v \,\rangle : \|v\| = 1\} \;\leq\; \sup\{\langle\, Av, v \,\rangle : \|v\| = 1\} \;\leq\; \lambda_1.$$

[Hint: For $x \in \mathbb{R}^n$ and $1 \leq k \leq n$, the function $g(x) = \sum_{i=1}^{k} x_i$ is Schur convex.]

References

Books

BECKENBACH, E., AND BELLMAN, R., *An Introduction to Inequalities,* The Mathematical Association of America, 1961 (Volume 3 in the New Mathematical Library).

A brief introduction to a handful of classical inequalities. Easy to read and includes full solutions to the exercises.

CLOUD, M. J., AND DRACHMAN, B. C., *Inequalities: With Applications to Engineering,* Springer-Verlag, 1998.

A brief introduction but a relatively broad selection of topics given its size. Includes sections on inner product spaces and matrix inequalities.

GARLING, D. J. H., *Inequalities: A Journey into Linear Analysis,* Cambridge University Press, 2007.

Highly advanced and very detailed.

HARDY, G. H., LITTLEWOOD, J. E., AND PÓLYA, G., *Inequalities,* Cambridge University Press, 2nd. ed., 1952.

The bible. But it's beginning to show its age and can be very hard to read in spots. Still a worthwhile reference for any mathematician, in spite of this minor shortcoming.

HORN, ROGER A., AND JOHNSON, CHARLES R., *Matrix Analysis,* Cambridge University Press, 1985; reprinted 1990.

Advanced and extremely thorough; the first of two volumes.

KAZARINOFF, N., *Analytic Inequalities,* Dover Publications, 2003 (originally published in 1961 by Holt, Rinehart, and Winston).

A brief introduction that relies heavily on tools from calculus. Its brevity is somewhat compensated for by the wide variety of exercises.

MARCUS, M. AND MINC, H., *A Survey of Matrix Theory and Matrix Inequalities,* Dover Publications, 1992 (originally published in 1969 by Prindle, Weber & Schmidt).

The emphasis here is on survey: Somewhat telegraphic, and hard to read in spots, but packed with information.

MARSHALL, A. W., AND OLKIN, I., *Inequalities: Theory of Majorization and its Applications,* Academic Press, 1979.

An excellent reference for applications of majorization to probability and statistics.

STEELE, J. MICHAEL, *The Cauchy-Schwarz Master Class: An Introduction to the Art of Mathematical Inequalities,* Cambridge University Press, 2004.

As much an honors seminar as a textbook. Steele approaches each new inequality as a "challenge problem." Full of interesting extensions, asides, and fodder for further study, but don't expect a formal textbook. Includes solutions to the exercises.

WEBSTER, ROGER, *Convexity,* Oxford University Press, 1994.

Very detailed and covers a wide range of topics, including sections on the Gamma function, Birkhoff's theorem, and more. In spite of some rough edges, it is an excellent reference. Far more detail than most books at this level. Includes solutions to the exercises.

Articles

Apostol, T. M., An elementary view of Euler's summation formula, *The American Mathematical Monthly,* **106** (1999), 409–418.

Carleson, L., A proof of an inequality of Carleman, *Proceedings of the American Mathematical Society,* **5** (1954), 932–933.

Duncan, J., and McGregor, C. M., Carleman's inequality, *The American Mathematical Monthly,* **110** (2003), 424–431.

Hurlbert, G., A short proof of the Birkhoff-von Neumann theorem, preprint (unpublished), 2008.

Impens, C., Stirling's series made easy, *The American Mathematical Monthly,* **110** (2003), 730–735.

Redheffer, R., and Volkmann, P., Schur's generalization of Hilbert's inequality, *Proceedings of the American Mathematical Society,* **87** (1983), 567–568.

Romik, D., Stirling's approximation for $n!$: the ultimate short proof, *The American Mathematical Monthly,* **107** (2000), 556–557.